Cylindrical Waveguides

by Dr. Colton, Physics 442 (last updated: Winter 2020)

Background

To consider the case of cylindrical waveguides, i.e. formed by a hollow cylinder of radius R, we again assume that the z- and t-dependence will be given by $e^{i(kz-\omega t)}$. This leads to the same result from the wave equation as with a rectangular waveguide, only expressed in cylindrical coordinates. The equations for E_z and B_z are therefore as follows:

$$\frac{\partial^2 E_z}{\partial s^2}+\frac{1}{s}\frac{\partial E_z}{\partial s}+\frac{1}{s^2}\frac{\partial^2 E_z}{\partial \varphi^2}+\left(\frac{\omega^2}{c^2}-k^2\right)E_z=0$$

$$\frac{\partial^2 B_z}{\partial s^2}+\frac{1}{s}\frac{\partial B_z}{\partial s}+\frac{1}{s^2}\frac{\partial^2 B_z}{\partial \varphi^2}+\left(\frac{\omega^2}{c^2}-k^2\right)B_z=0$$

TM modes, Separation of Variables

For the TM modes, we have $B_z = 0$ and $E_z \neq 0$. We therefore focus on E_z. Using the separation of variables technique, we assume that the solution has the form $E_z = S(s)\Phi(\phi)$. This turns the equation for E_z into:

$$S''\Phi+\frac{1}{s}S'\Phi+\frac{1}{s^2}S\Phi''+\left(\frac{\omega^2}{c^2}-k^2\right)S\Phi=0$$

Dividing both sides by $S\Phi$ and multiplying by s^2, we have:

$$s^2\frac{S''}{S}+s\frac{S'}{S}++\frac{\Phi''}{\Phi}+s^2\left(\frac{\omega^2}{c^2}-k^2\right)=0$$

Now bring the ϕ term over to the right hand side, and we have successfully separated the variables.

$$s^2\frac{S''}{S}+s\frac{S'}{S}+s^2\left(\frac{\omega^2}{c^2}-k^2\right)=-\frac{\Phi''}{\Phi}$$

The left hand side is just a function of s, the right hand side is just a function of ϕ, so they can only be equal if they both equal a constant. It could be a positive or a negative constant, but because I know the answer I will guess correctly and make it a positive constant. To enforce that, we set it equal to α^2.

$$s^2\frac{S''}{S}+s\frac{S'}{S}+s^2\left(\frac{\omega^2}{c^2}-k^2\right)=-\frac{\Phi''}{\Phi}=\alpha^2$$

This is actually two equations, one for s and one for ϕ.

$$s^2\frac{S''}{S}+s\frac{S'}{S}+s^2\left(\frac{\omega^2}{c^2}-k^2\right)=\alpha^2$$

$$\frac{\Phi''}{\Phi} = -\alpha^2$$

Solving the Φ equation

Let's solve the ϕ equation first. It's easy! $\Phi'' = -\alpha^2\Phi$ means that

$$\Phi = \begin{cases} \sin \alpha\phi \\ \cos \alpha\phi \end{cases}$$

or linear combinations.

Because $\Phi(\phi)$ and $\Phi(\phi + 2\pi)$ need to give the same value, this gives an added constraint that:

$$\boxed{\alpha = \text{integer}}$$

If that's not obvious to you, try for example setting $\phi = 30°$ and comparing $\sin(\alpha(30°))$ to $\sin(\alpha(30° + 360°))$ when α is not an integer.

We can rotate the x- and y-axes such that we only get the cosine function. End result for ϕ, not including an arbitrary amplitude:

$$\boxed{\Phi = \cos \alpha\phi}$$

Solving the *S* equation

Now back to the S equation…

$$s^2\frac{S''}{S} + s\frac{S'}{S} + s^2\left(\frac{\omega^2}{c^2} - k^2\right) = \alpha^2$$

$$s^2S'' + sS' + S\left(s^2\left(\frac{\omega^2}{c^2} - k^2\right) - \alpha^2\right)$$

Consider the term $\frac{\omega^2}{c^2} - k^2$. It has units of (1/length)^2. By multiplying it by R^2, we can turn it into a dimensionless number. For reasons that will soon become clear, I'll call that number $u_{\alpha m}^2$, so

$$u_{\alpha m} = R\sqrt{\frac{\omega^2}{c^2} - k^2}$$

That also means that:

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{u_{\alpha m}^2}{R^2}}$$

Plugging that substitution for $u_{\alpha m}$ back into the S equation, it turns the equation into this:

$$s^2 S'' + sS' + S\left(\left(\frac{u_{\alpha m} s}{R}\right)^2 - \alpha^2\right) = 0$$

This is Bessel's equation! Written for $x = \frac{u_{\alpha m} s}{R}$. (Here x is a dimensionless variable, not the x-coordinate.)

Its solutions are the Bessel functions:

$$S = \begin{cases} J_\alpha(x) \\ Y_\alpha(x) \end{cases}$$

or linear combinations.

The $J_\alpha(x)$ functions are the regular Bessel functions. The $Y_\alpha(x)$ functions are the "Bessel functions of the second kind", which go infinite at the origin ($s = 0$). Since we don't want solutions which are infinte at the origin, we through them out, leaving us the end result for s, not including an arbitrary amplitude:

$$S = J_\alpha\left(\frac{u_{\alpha m} s}{R}\right)$$

Putting together ϕ and s solutions

Putting the solutions together, $E_z = S(s)\Phi(\phi)$, and still not worrying about an arbitrary amplitude, the answer is therefore:

$$E_z = J_\alpha\left(\frac{u_{\alpha m} s}{R}\right) \cos \alpha\phi$$

There could also be a summation over α; however usually people just consider each α separately, as done below.

Boundary conditions

The governing boundary conditions are these two, evaluated at $s = R$.

$$E_{//} = 0$$

$$B_\perp = 0$$

For the TM modes, we must focus on the $E_{//}$ boundary condition. E_z is actually the parallel component, so it means $E_z = 0$.

$$J_\alpha\left(\frac{u_{\alpha m}R}{R}\right)\cos\alpha\phi = 0$$

$$J_\alpha(u_{\alpha m})\cos\alpha\phi = 0$$

We see now that the $u_{\alpha m}$ values defined earlier must be the zereos of the Bessel functions.

Some sample modes

<u>$\alpha = 0$</u>

$$E_z = J_0\left(\frac{u_{0m}s}{R}\right)$$

E_z has no ϕ dependence. m can be any integer, and u_{0m} is the mth zero of the J_0 Bessel function. The possible modes are TM_{01}, TM_{02}, TM_{03}, etc.

<u>$\alpha = 1$</u>

$$E_z = J_1\left(\frac{u_{1m}s}{R}\right)\cos\phi$$

E_z does have ϕ dependence now. m can be any integer, and u_{1m} is the mth zero of the J_1 Bessel function. The possible modes are TM_{11}, TM_{12}, TM_{13}, etc.

Hopefully extrapolations to higher α values are clear.

Conclusion

The allowed TM modes ($B_z = 0$) are chacterized by integer values for α and m. For a given mode, $E_z = J_\alpha\left(\frac{u_{\alpha m}s}{R}\right)\cos\alpha\phi$. From E_z, one can deduce all of the other components of the electric and magnetic fields using the "longitudinal to transverse" equations, if one desires. And the dispersion equation of a given mode is given by:

$$k = \sqrt{\frac{\omega^2}{c^2} - \frac{u_{\alpha m}^2}{R^2}}$$

where (for the TM modes), $u_{\alpha m}$ is the mth zero of the J_α Bessel function.