Poynting's Theorem

by Dr. Colton, Physics 442/471 (last updated: Summer 2021)

Derivation

Beginning with Maxwell's equations…

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(dot with } \mathbf{B}) \rightarrow \mathbf{B} \cdot (\nabla \times \mathbf{E}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(dot with } \mathbf{E}) \rightarrow \mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mu_0 \mathbf{E} \cdot \mathbf{J} + \epsilon_0 \mu_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{B} \cdot (\nabla \times \mathbf{E}) = -\frac{1}{2} \frac{\partial (B^2)}{\partial t}$$

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mu_0 \mathbf{E} \cdot \mathbf{J} + \frac{\epsilon_0 \mu_0}{2} \frac{\partial (E^2)}{\partial t}$$

Bottom equation minus top equation…

$$\underbrace{\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E})}_{-\nabla \cdot (\mathbf{E} \times \mathbf{B})} = \mu_0 \mathbf{E} \cdot \mathbf{J} + \underbrace{\frac{\epsilon_0 \mu_0}{2} \frac{\partial (E^2)}{\partial t} + \frac{1}{2} \frac{\partial (B^2)}{\partial t}}_{\mu_0 \frac{\partial}{\partial t}\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)}$$

Multiply both sides by $-1/\mu_0$…

$$\nabla \cdot \left(\frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \right) = -\mathbf{E} \cdot \mathbf{J} - \frac{\partial}{\partial t} \left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right)$$

There are three terms to discuss here. The left side of the equation is: $\nabla \cdot \left(\frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B}) \right)$. The stuff inside the divergence operator is the energy per unit time, per area, transported by the fields; define it with the symbol **S** called the **Poynting vector**.

The second term, $-\mathbf{E} \cdot \mathbf{J}$, is the rate at which work is done by the fields, per volume, on the flowing charges in the medium which comprise **J**. This comes from the Lorentz force law, and is derived in Griffiths (442 textbook) just above Eq. 8.6. Peatross & Ware (471 textbook) call it $\frac{\partial u_{medium}}{\partial t}$ and Griffiths doesn't really have a term for it but might call it lower case $\frac{\partial w}{dt}$ since its volume integral is capital $\frac{\partial W}{\partial t}$. I'll use that notation.

Finally, the last term, $-\frac{\partial}{\partial t}\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)$, is the energy stored in the **E** and **B** fields, per volume; P&W call it u_{field} and Griffiths calls it more simply, u.

Conclusion

The equation with the three terms is Poynting's Theorem; written with the terms slightly rearranged it is:

$$\boxed{-\frac{\partial u}{\partial t} = \frac{\partial w}{\partial t} + \nabla \cdot \mathbf{S}}$$

In integral form where U represents the total energy stored in the fields in some volume of space and W is the work done by the fields on the charges inside that volume, it looks like this:

$$\boxed{-\frac{\partial U}{\partial t} = \frac{\partial W}{\partial t} + \oint \mathbf{S} \cdot d\mathbf{A}}$$

(I used the divergence theorem to change the volume integral of $\nabla \cdot \mathbf{S}$ into the integral of $\mathbf{S} \cdot d\mathbf{A}$ over the surface bounding the volume.)

It is a statement of conservation of energy. If energy which has been stored in the fields is lost ($\partial U/\partial t$ is negative, which makes the left hand side positive), then that energy either goes into doing work on the charges in that volume of space, or else it gets transported out of the volume. The rightmost term can be thought of as an energy flux (per time).

In a region of space where there is no medium, it becomes:

$$-\frac{\partial u}{\partial t} = \nabla \cdot \mathbf{S}$$

Compare that statement of energy conservation to the equation of continuity, which is a statement of charge conservation; you can see they have the identical mathematical form:

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J}$$