Bessel Functions, by Dr Colton
Physics 442, Winter 2017

The Bessel functions, $J_\alpha(x)$ are a series of functions, that:

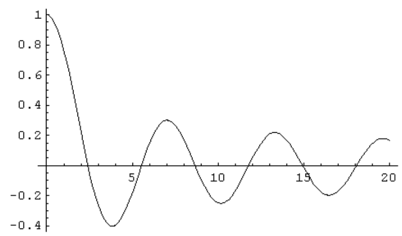
(a) come up often, especially in partial differential equations
(b) have interesting properties
(c) are well understood and have been studied for centuries

They are typically only used for positive values of x. Here are the first four plotted.

Plots

BesselJ[α,x] is a built-in *Mathematica* function just like Sin[x], and gives $J_\alpha(x)$

In[25]:= `Plot[BesselJ[0, x], {x, 0, 20}]`

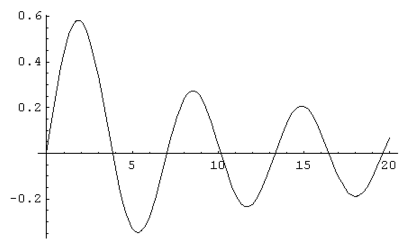


$J_0(x)$
crosses zero at 2.405, 5.520, 8.654, …

← The only one that is not zero at the origin

Out[25]= `- Graphics -`

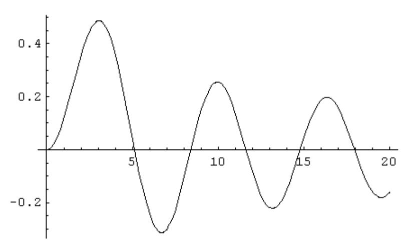
In[26]:= `Plot[BesselJ[1, x], {x, 0, 20}]`



$J_1(x)$
crosses zero at 3.832, 7.016, 10.173, …

Out[26]= `- Graphics -`

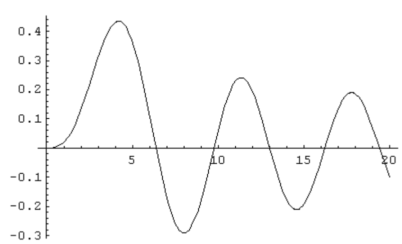
In[27]:= `Plot[BesselJ[2, x], {x, 0, 20}]`



$J_2(x)$
crosses zero at 5.136, 8.417, 11.620, …

Out[27]= `- Graphics -`

In[28]:= `Plot[BesselJ[3, x], {x, 0, 20}]`



$J_3(x)$
crosses zero at 6.380, 9.761, 13.015, …

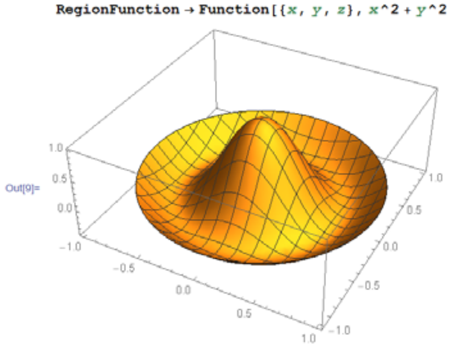
Out[28]= `- Graphics -`

Very important facts

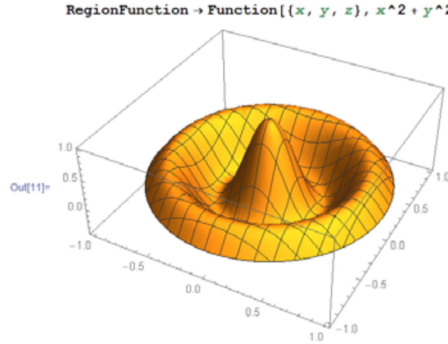
- Can be computed via a series formula: $J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\alpha}}{k!(k+\alpha)!2^{2k+\alpha}}$.
- Derivatives:
    - $\frac{d}{dx}\big(J_0(x)\big) = -J_1(x)$ (which means the max/min of J_0 are at the zeroes of J_1)
    - $\frac{d}{dx}\big(J_n(x)\big) = \frac{J_{n-1}(x) - J_{n+1}(x)}{2}$
- Zeroes: $u_{\alpha m}$ represents the m^{th} zero of $J_\alpha(x)$. From the previous page:
    - $u_{01} = 2.405$, $u_{02} = 5.520$, $u_{03} = 8.654$, …
    - $u_{11} = 3.832$, $u_{12} = 7.016$, $u_{13} = 10.173$, …
    - etc.

  These numbers are available in Mathematica via the BesselJZero function: BesselJZero[0,1] yields a result of 2.4048255576957727686…, etc.
- Differential equations ("Bessel's equation"):
    - $x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + (x^2 - \alpha^2) f = 0$ has solution $J_\alpha(x)$, or linear combinations
    - $x^2 \frac{d^2 f}{dx^2} + x \frac{df}{dx} + ({u_{\alpha m}}^2 x^2 - \alpha^2) f = 0$ has solution $J_\alpha(u_{\alpha m} x)$, or linear combinations
- There are a second set of solutions to Bessel's differential equation, called $Y_\alpha(x)$ (sometimes written $N_\alpha(x)$) or the "Bessel functions of the second kind"; these diverge at $x = 0$ and are therefore often not used.
    - Various other linear combinations of $J_\alpha(x)$ and $Y_\alpha(x)$ are sometimes used; two examples are the "modified Bessel functions" and the "Hankel functions". These are beyond the scope of this course.
    - The so-called "spherical Bessel functions" (and "spherical Hankel functions") are solutions to a different, albeit closely related, differential equation. These are also beyond the scope of this course.
- When $u_{\alpha m}$ is added to the argument in the form $u_{\alpha m} r$, the functions go to zero at $r = 1$. Then e.g. $J_0(u_{0m} r)$ represents a series of functions between the origin and a circular boundary of $r = 1$, having m antinodes. For example $J_0(u_{02} x)$ has two antinodes, $J_0(u_{03} x)$ has three antinodes, etc.

```
In[8]:= u02 = BesselJZero[0, 2];
        Plot3D[BesselJ[0, u02 Sqrt[x^2 + y^2]], {x, -1, 1}, {y, -1, 1},
         RegionFunction → Function[{x, y, z}, x^2 + y^2 < 1]]
```



```
In[10]:= u03 = BesselJZero[0, 3];
         Plot3D[BesselJ[0, u03 Sqrt[x^2 + y^2]], {x, -1, 1}, {y, -1, 1},
          RegionFunction → Function[{x, y, z}, x^2 + y^2 < 1]]
```



- Orthogonality: $J_0(u_{0m} x)$ and $J_0(u_{0n} x)$ are orthogonal over the interval (0, 1) with respect to a weighting function of x:
    - $\int_0^1 x J_0(u_{0m} x) J_0(u_{0n} x) dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2}\big(J_1(u_{0m})\big)^2 & \text{if } n = m \end{cases}$

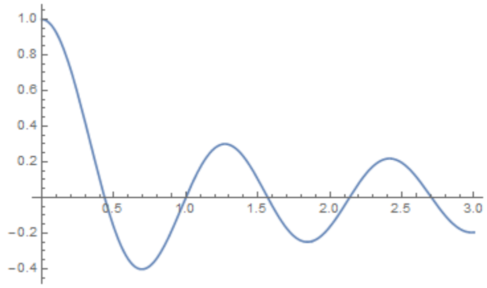
  Or, more generally for $J_\alpha(x)$:
    - $\int_0^1 x J_\alpha(u_{\alpha m} x) J_\alpha(u_{\alpha n} x) dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2}\big(J_{\alpha+1}(u_{\alpha m})\big)^2 & \text{if } n = m \end{cases}$

Orthogonality, depicted

```
In[1]:= u02 = BesselJZero[0, 2];
        u03 = BesselJZero[0, 3];
```

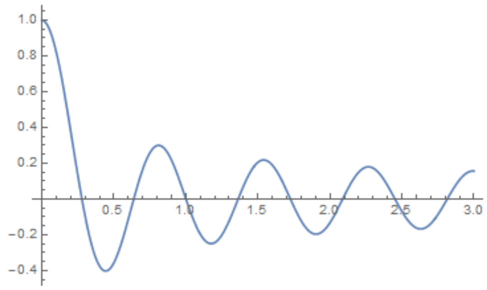
```
In[3]:= Plot[BesselJ[0, u02 x], {x, 0, 3}]
```

Out[3]=



```
In[4]:= Plot[BesselJ[0, u03 x], {x, 0, 3}, PlotRange → All]
```

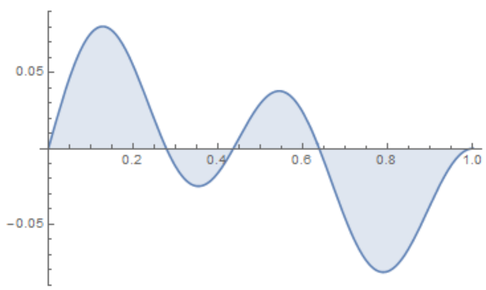
Out[4]=



These are two functions which are orthogonal. They are both J_0 functions, but one has been scaled so that its *second* "zero crossing" occurs at $x = 1$, and the other has been scalled so that its *third* zero crossing occurs at $x = 1$.

```
In[5]:= Plot[x BesselJ[0, u02 x] BesselJ[0, u03 x], {x, 0, 1}, Filling → Axis]
```

Out[5]=



The two functions are only orthogonal over the range from 0 to 1, and only with a weighting function of x. But within those constraints, you can see that the positive areas cancel out the negative areas.

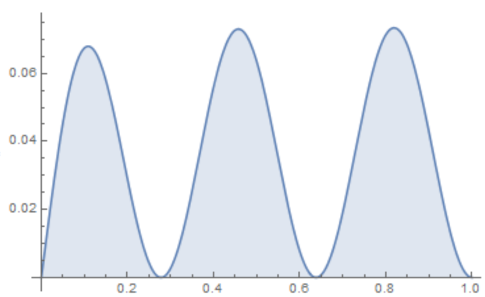
The integral is zero.

```
In[6]:= Integrate[x BesselJ[0, u02 x] BesselJ[0, u03 x], {x, 0, 1}]
Out[6]= 0
```

$\int_0^1 xJ_0(u_{0m}x)J_0(u_{0n}x)dx$ equals 0 if $n \neq m$!

```
In[7]:= Plot[x BesselJ[0, u03 x] BesselJ[0, u03 x], {x, 0, 1}, Filling → Axis]
```

Out[7]=



By contrast, $J_0(u_{0m}x)$ is not orthogonal to itself.

```
In[8]:= Integrate[x BesselJ[0, u03 x] BesselJ[0, u03 x], {x, 0, 1}] // N
Out[8]= 0.0368432

In[9]:= .5 (BesselJ[1, u03]) ^2 // N
Out[9]= 0.0368432
```

$\int_0^1 xJ_0(u_{0m}x)J_0(u_{0n}x)dx$ equals $\frac{1}{2}\left(J_1(u_{0m})\right)^2$ if $n = m$!

Comparison with sines & cosines:

	Sines/Cosines	**Bessel functions**
1.	Two oscillatory functions: sin(x) and cos(x). Sometimes one of them is not used, due to the symmetry of the problem.	Two oscillatory functions for each α: $J_\alpha(x)$ and $Y_\alpha(x)$. Typically Y_α is not used because it's infinite at the origin.
2.	You determine the value of sin(x) or cos(x) for arbitrary x by using a calculator or computer program.	You determine the value of $J_\alpha(x)$ for arbitrary x by using a calculator or computer program.
3.	$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\alpha}}{k!(k+\alpha)!2^{2k+\alpha}}$
	<u>Consider just sin(x):</u>	<u>Consider just $J_\alpha(x)$ for one α, say $\alpha = 0$:</u> (similar things hold true for all α's)
4.	The zeroes of sin(x) are at $x = \pi, 2\pi, 3\pi$, etc. $x =$ "$m\pi$" is the m^{th} zero	The zeroes of $J_0(x)$ are at $x \approx 2.405, 5.520, 8.654$, etc. $x =$ "u_{0m}" is the m^{th} zero
5.	Using the zeroes in the argument, $\sin(m\pi x)$ has m antinodes in the interval from 0 to 1. And at $x = 1$ (the boundary), $\sin(m\pi x) = 0$ for all m.	Using the zeroes in the argument, $J_0(u_{0m}r)$ has m antinodes in the circular region $r \le 1$. And at $r = 1$ (the boundary), $J_0(u_{0m}x) = 0$ for all m.
6.	The differential equation satisfied by $f = \sin(x)$ is $f'' + f = 0$. The differential equation satisfied by $f = \sin(m\pi x)$ is $f'' + (m\pi)^2 f = 0$.	The differential equation satisfied by $f = J_0(x)$ is $x^2 f'' + x f' + (x^2 - 0^2) f = 0$. The differential equation satisfied by $f = J_0(u_{0m}x)$ is $x^2 f'' + x f' + (u_{0m}^2 x^2 - 0^2) f = 0$. $0^2 \to \alpha^2$ for other α's
7.	$\sin(n\pi x)$ is orthogonal to $\sin(m\pi x)$ on the interval (0,1): $\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} 0, if\ n \neq m \\ \frac{1}{2}, if\ n = m \end{cases}$	$J_0(u_{0n}x)$ is orthogonal to $J_0(u_{0m}x)$ on the interval (0,1), with respect to a weighting function of x: $\int_0^1 x J_0(u_{0n}x) J_0(u_{0m}x)dx = \begin{cases} 0, if\ n \neq m \\ \frac{1}{2}\left(J_1(u_{0m})\right)^2, if\ n = m \end{cases}$ Similar orthogonality holds for other values of α.

Additionally, the Bessel functions are related to sines/cosines through this integral formula:

$$J_\alpha(x) = \frac{1}{\pi}\int_0^\pi \cos(\alpha\theta - x\sin\theta)d\theta$$

Quote from Mary Boas, in *Mathematical Methods in the Physical Sciences*: "In fact, if you had first learned about sin(nx) and cos(nx) as power series solutions of $y'' = -n^2 y$, instead of in elementary trigonometry, you would not feel that Bessel functions were appreciably more difficult or strange than trigonometric functions. Like sines and cosines, Bessel functions are solutions of a differential equation; they are tabulated and their graphs can be drawn; they can be represented as a series; and a large number of formulas about them are known."