Six Derivations, by Dr. Colton Winter 2018

In the derivations that follow, $\boldsymbol{\mathscr{r}} = \mathbf{r} - \mathbf{w}(t_r)$, where $\mathbf{w}(t)$ is the particle's trajectory and t_r is the retarded time.

Derivation 1

$$\left(\frac{\partial \mathscr{r}}{\partial x}\right)_{y,z,t,t_r} = \frac{\partial}{\partial x}\left(\sqrt{(x-w_x)^2 + \left(y-w_y\right)^2 + (z-w_z)^2}\right)_{y,z,t,t_r}$$

y and *z* are explicitly held constant,
and since *t* and t_r are held constant ***w*** will also be fixed.

$$\left(\frac{\partial \mathscr{r}}{\partial x}\right)_{y,z,t,t_r} = \frac{1}{2}\left(\sqrt{(x-w_x)^2 + \left(y-w_y\right)^2 + (z-w_z)^2}\right)^{-\frac{1}{2}} \cdot 2(x-w_x)$$

$$= \frac{1}{\mathscr{r}} \cdot \mathscr{r}_x$$

$$\boxed{\left(\frac{\partial \mathscr{r}}{\partial x}\right)_{y,z,t,t_r} = \frac{\mathscr{r}_x}{\mathscr{r}}}$$

Derivation 2

$$\left(\frac{\partial(\mathbf{v} \cdot \boldsymbol{\mathscr{r}})}{\partial x}\right)_{y,z,t,t_r} = \frac{\partial}{\partial x}\left(v_x\mathscr{r}_x + v_y\mathscr{r}_y + v_z\mathscr{r}_z\right)_{y,z,t,t_r}$$

v is a constant, so just need derivatives of $\mathscr{r}_x, \mathscr{r}_y, \mathscr{r}_z$.
However, $\mathscr{r}_y$ and $\mathscr{r}_y$ have no *x* dependence so those terms go to zero.

$$\left(\frac{\partial(\mathbf{v} \cdot \boldsymbol{\mathscr{r}})}{\partial x}\right)_{y,z,t,t_r} = v_x \frac{\partial \mathscr{r}_x}{\partial x}$$

$$= v_x \frac{\partial(x-w_x)}{\partial x}$$

$$\boxed{\left(\frac{\partial(\mathbf{v} \cdot \boldsymbol{\mathscr{r}})}{\partial x}\right)_{y,z,t,t_r} = v_x}$$

Derivation 3

$$\left(\frac{\partial \mathscr{r}}{\partial t_r}\right)_{x,y,z,t} = \frac{\partial}{\partial t_r}\left(\sqrt{(x-w_x)^2 + \left(y-w_y\right)^2 + (z-w_z)^2}\right)_{x,y,z,t}$$

The *w*'s are variables, the others are constants.

$$\left(\frac{\partial \mathscr{r}}{\partial t_r}\right)_{x,y,z,t} = \frac{1}{2}(\ldots)^{-\frac{1}{2}} \cdot \left(2(x-w_x)\left(-\frac{\partial w_x}{\partial t_r}\right) + 2\left(y-w_y\right)\left(-\frac{\partial w_y}{\partial t_r}\right) + 2(z-w_z)\left(-\frac{\partial w_z}{\partial t_r}\right)\right)$$

$$= \frac{1}{\mathscr{r}} \cdot \left(\mathscr{r}_x(-v_x) + \mathscr{r}_y(-v_y) + \mathscr{r}_z(-v_z) \right)$$

$$= \frac{1}{\mathscr{r}} \cdot \left(\boldsymbol{\mathscr{r}} \cdot (-\mathbf{v}) \right)$$

$$\boxed{\left(\frac{\partial \mathscr{r}}{\partial t_r}\right)_{x,y,z,t} = -\frac{\boldsymbol{\mathscr{r}} \cdot \mathbf{v}}{\mathscr{r}}}$$

Derivation 4

$$\left(\frac{\partial(\mathbf{v} \cdot \boldsymbol{\mathscr{r}})}{\partial t_r}\right)_{x,y,z,t} = \frac{\partial}{\partial t_r}\left(v_x \mathscr{r}_x + v_y \mathscr{r}_y + v_z \mathscr{r}_z\right)_{x,y,z,t}$$

Use the product rule on each term.

$$\left(\frac{\partial(\mathbf{v} \cdot \boldsymbol{\mathscr{r}})}{\partial t_r}\right)_{x,y,z,t} = v_x \frac{\partial \mathscr{r}_x}{\partial t_r} + \mathscr{r}_x \frac{\partial v_x}{\partial t_r} + \cdots$$

$$= v_x \frac{\partial (x - w_x)}{\partial t_r} + \mathscr{r}_x a_x + \cdots$$

$$= v_x(-v_x) + \mathscr{r}_x a_x + \cdots$$

$$= -v_x^2 + \mathscr{r}_x a_x + \cdots$$

Add back in the other four terms.

$$\left(\frac{\partial(\mathbf{v} \cdot \boldsymbol{\mathscr{r}})}{\partial t_r}\right)_{x,y,z,t} = -v_x^2 + \mathscr{r}_x a_x - v_y^2 + \mathscr{r}_y a_y - v_z^2 + \mathscr{r}_z a_z$$

$$\boxed{\left(\frac{\partial(\mathbf{v} \cdot \boldsymbol{\mathscr{r}})}{\partial t_r}\right)_{x,y,z,t} = -v^2 + \boldsymbol{\mathscr{r}} \cdot \mathbf{a}}$$

Derivation 5 (the hardest one!)

$$\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t} = \frac{\partial}{\partial x}\left(t - \frac{\mathscr{r}}{c}\right)_{y,z,t}$$

t is independent of *x*.

$$\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t} = -\frac{1}{c}\left(\frac{\partial \mathscr{r}}{\partial x}\right)_{y,z,t}$$

Use the chain rule.

$$\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t} = -\frac{1}{c}\left(\frac{\partial \mathscr{r}}{\partial x}\right)_{y,z,t,t_r} - \frac{1}{c}\left(\frac{\partial \mathscr{r}}{\partial t_r}\right)_{x,y,z,t}\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t}$$

First term: use Derivation 1.
Second term: use Derivation 3.

$$\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t} = -\frac{1}{c}\frac{\mathscr{r}_x}{\mathscr{r}} + \frac{1}{c}\frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{\mathscr{r}}\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t}$$

Move the $\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t}$ term that's on the RHS to the left, then factor, then divide.

$$\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t} - \frac{1}{c}\frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{\mathscr{r}}\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t} = -\frac{1}{c}\frac{\mathscr{r}_x}{\mathscr{r}}$$

$$\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t}\left(1 - \frac{1}{c}\frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{\mathscr{r}}\right) = -\frac{1}{c}\frac{\mathscr{r}_x}{\mathscr{r}}$$

$$\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t} = -\frac{\mathscr{r}_x}{\mathscr{r}\,c}\frac{1}{\left(1 - \frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{\mathscr{r}\,c}\right)}$$

$$\boxed{\left(\frac{\partial t_r}{\partial x}\right)_{y,z,t} = -\frac{\mathscr{r}_x}{\mathscr{r}\,c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}}$$

Derivation 6 (a lot like Derivation 5)

$$\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z} = \frac{\partial}{\partial t}\left(t - \frac{\mathscr{r}}{c}\right)_{x,y,z}$$

$$\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z} = 1 - \frac{1}{c}\left(\frac{\partial \mathscr{r}}{\partial t}\right)_{x,y,z}$$

Use the chain rule.

$$\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z} = 1 - \frac{1}{c}\left(\frac{\partial \mathscr{r}}{\partial t_r}\right)_{x,y,z,t}\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z}$$

Second term: use Derivation 3.

$$\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z} = 1 + \frac{1}{c}\frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{\mathscr{r}}\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z}$$

Move the $\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z}$ term that's on the RHS to the left, then factor, then divide.

$$\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z} - \frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{\mathscr{r}\,c}\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z} = 1$$

$$\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z}\left(1 - \frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{\mathscr{r}\,c}\right) = 1$$

$$\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z} = -\frac{1}{\left(1 - \frac{\boldsymbol{\mathscr{r}}\cdot\mathbf{v}}{\mathscr{r}\,c}\right)}$$

$$\boxed{\left(\frac{\partial t_r}{\partial t}\right)_{x,y,z} = -\frac{\mathscr{r}\,c}{\mathscr{r}\,c - \boldsymbol{\mathscr{r}}\cdot\mathbf{v}}}$$