Lorentz Model: Driven Damped Harmonic Oscillator, by Dr. Colton
Physics 471 – Optics

"Lorentz model": charge on a spring moving back & forth, a "driven damped harmonic oscillator"
…driven by oscillating electric field $E = E_0\cos(-\omega t)$
→ I'm using $\cos(-\omega t)$ instead of $\cos(+\omega t)$ so it matches time dependence of standard traveling EM wave, $\cos(kx-\omega t)$
…with velocity-dependent damping described by γ (units of γ chosen such that force = γmv)

$\sum F = m\ddot{x}$
$F_{driving} + F_{spring} + F_{damping} = m\ddot{x}$
$qE_0\cos(-\omega t) - kx - \gamma m\dot{x} - m\ddot{x}$
$\ddot{x} + \gamma\dot{x} + \frac{k}{m}x = \frac{qE_0}{m}\cos(-\omega t)$ → Let $\omega_0 = \sqrt{\frac{k}{m}}$

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{qE_0}{m}\cos(-\omega t)}$$ This is the equation of motion we need to solve

Guess $x = x_0\cos(-\omega t + \phi)$ as solution → $\tilde{x} = x_0 e^{i\phi}e^{-i\omega t}$
$\tilde{x} = \tilde{x}_0 e^{-i\omega t}$ (ϕ is lumped in with complex $\tilde{x}_0$)

Derivatives: $\dot{\tilde{x}} = (-i\omega)\tilde{x}_0 e^{-i\omega t}$
$\ddot{\tilde{x}} = (-i\omega)^2\tilde{x}_0 e^{-i\omega t}$

Plug into boxed equation, also convert cosine into complex exponential:
$(-i\omega)^2\tilde{x}_0 e^{-i\omega t} + \gamma(-i\omega)\tilde{x}_0 e^{-i\omega t} + \omega_0^2\tilde{x}_0 e^{-i\omega t} = \frac{qE_0}{m}e^{-i\omega t}$

Cancel the $e^{-i\omega t}$ factors:
$\tilde{x}_0(-\omega^2 - i\omega\gamma + \omega_0^2) = \frac{qE_0}{m}$

$$\boxed{\tilde{x}_0 = \frac{qE_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}}$$ The answer in compact form!!

Or with explicit time dependence:
$\tilde{x} = \frac{qE_0}{m}\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}e^{-i\omega t}$

What does it mean? To understand better, we could convert to polar form. Need to first get real & imaginary parts.

Work with this term:
$$\frac{1}{\omega_0^2-\omega^2-i\omega\gamma}\times\frac{\omega_0^2-\omega^2+i\omega\gamma}{\omega_0^2-\omega^2+i\omega\gamma} = \frac{\omega_0^2-\omega^2+i\omega\gamma}{\left(\omega_0^2-\omega^2\right)^2+(\omega\gamma)^2}$$
$$= \frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+(\omega\gamma)^2} + i\frac{\omega\gamma}{\left(\omega_0^2-\omega^2\right)^2+(\omega\gamma)^2}$$ (now in real + imaginary form)

Real part of $\tilde{x}_0 = \frac{qE_0}{m}\frac{\omega_0^2-\omega^2}{\left(\omega_0^2-\omega^2\right)^2+(\omega\gamma)^2}$

Imaginary part of $\tilde{x}_0 = \frac{qE_0}{m}\frac{\omega\gamma}{\left(\omega_0^2-\omega^2\right)^2+(\omega\gamma)^2}$

Could convert to polar form: $\sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}\angle\phi$

Then what this all means, since $\tilde{x}_0$ represents the amplitude and phase of the oscillation, is that:
$x(t) = \sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}\cos(-\omega t + \phi)$

Much of this was really just algebra… the physics was finished after the boxed equation for $\tilde{x}_0$ labeled "the answer in compact form" on the previous page.