Complex wave number, $\tilde{k}$

by Dr. Colton, Physics 442 (last updated: Winter 2020)

Notation

When we start talking about *complex wave numbers*, we necessarily also have *complex indices of refraction* and *complex dielectric constants*, and there is a dizzying amount of varying notation that people use for those three quantities. To be as clear as possible and avoid any ambiguity, I myself will use the following symbols in class and on homework assignments and exams.

$$\tilde{k} = \text{complex wave vector}$$

$$k_{real} = \text{real part of } \tilde{k}, \text{called } k \text{ in Griffiths}$$

$$k_{imag} = \text{imaginary part of } \tilde{k}, \text{called } \kappa \text{ in Griffiths}$$

$$\tilde{n} = \text{complex index of refraction, not directly used in Griffiths,}$$
$$\text{also called } \mathcal{N} \text{ by some sources (e.g. Peatross and Ware)}$$

$$n_{real} = \text{real part of } \tilde{n}, \text{also called } n \text{ by nearly all sources (including Griffiths)}$$

$$n_{imag} = \text{imaginary part of } \tilde{n}, \text{not directly used in Griffiths,}$$
$$\text{also called } k \text{ by nearly all sources (!)}$$
$$\text{and } \kappa \text{ by some (e.g. Peatross and Ware)}$$

→ to add insult to injury, many sources define the index of refraction k as the *negative* of n_{imag}

$$\tilde{\epsilon}_r = \text{complex dielectric constant (aka relative permittivity),}$$
$$\text{also called } \epsilon \text{ by many sources and } K \text{ or } \tilde{\mathrm{K}} \text{ by others}$$

$$\epsilon_{r,real} = \text{real part of } \tilde{\epsilon}_r, \text{not directly used in Griffiths,}$$
$$\text{also called } \epsilon_1 \text{ or } \epsilon' \text{ by many sources and } K_{\mathrm{r}} \text{ or } K' \text{ by others}$$

$$\epsilon_{r,imag} = \text{imaginary part of } \tilde{\epsilon}_r, \text{not directly used in Griffiths,}$$
$$\text{also called } \epsilon_2 \text{ or } \epsilon'' \text{ by many sources and } K_{\mathrm{i}} \text{ or } K'' \text{ by others}$$

Basic relationships

The complex index of refraction $\tilde{n}$ and complex wave number $\tilde{k}$ are related through this equation:

$$\boxed{\tilde{n} = \frac{c}{\omega}\tilde{k}}$$

Equating the real and imaginary parts of that equation (and remembering that ω must be real) results in:

$$n_{real} = \frac{c}{\omega}k_{real}$$

$$n_{imag} = \frac{c}{\omega} k_{imag}$$

Note in particular that $\tilde{n}$ and $\tilde{k}$ will always have the same complex angle.

The complex dielectric constant $\tilde{\epsilon}_r$ and complex index of refraction $\tilde{n}$ are related through this equation:

$$\boxed{\tilde{\epsilon}_r = \tilde{n}^2}$$

Plugging in $\tilde{\epsilon}_r = \epsilon_{r,real} + i\epsilon_{r,imag}$ and $\tilde{n} = n_{real} + in_{imag}$ and equating the real and imaginary parts of that equation results in:

$$\epsilon_{r,real} = n_{real}^2 - n_{imag}^2$$

$$\epsilon_{r,imag} = 2n_{real}n_{imag}$$

(As per the "to add insult to injury" comment above, you will see many sources write that last equation as $\epsilon'' = -2nk$.)

The complex dielectric constant $\tilde{\epsilon}_r$ and complex wave number $\tilde{k}$ are related through this equation:

$$\boxed{\tilde{\epsilon}_r = \frac{c^2}{\omega^2}\tilde{k}^2}$$

Plugging in $\tilde{\epsilon}_r = \epsilon_{r,real} + i\epsilon_{r,imag}$ and $\tilde{k} = k_{real} + ik_{imag}$ and equating the real and imaginary parts of that equation results in:

$$\epsilon_{r,real} = \frac{c^2}{\omega^2}\left(k_{real}^2 - k_{imag}^2\right)$$

$$\epsilon_{r,imag} = \frac{c^2}{\omega^2}\left(2k_{real}k_{imag}\right)$$

Why is the complex wave number important?

The complex wave number governs the wavelength in the material as well as the decay of the wave's amplitude. Assuming travel in the z-direction for simplicity, plane waves have the following form:

$$\tilde{E} = E_0 e^{i(kz-\omega t)}$$

When the wave number is complex, this becomes:

$$\tilde{E} = E_0 e^{i(\tilde{k}z-\omega t)}$$

$$\tilde{E} = E_0 e^{i((k_{real}+ik_{imag})z-\omega t)}$$

$$\tilde{E} = E_0 e^{-k_{imag}z} e^{i(k_{real}z-\omega t)}$$

Taking the real part, we have:

$$E = E_0 e^{-k_{imag}z} \cos(k_{real}z - \omega t)$$

You can see from that, that the real part of $\tilde{k}$ is related to the oscillations in space as per the usual definition of wave number, through the $\cos(k_{real}z - \omega t)$ term:

$$\boxed{\lambda = \frac{2\pi}{k_{real}}}$$

And the imaginary part of $\tilde{k}$ is related to how the amplitude of the wave decays as it penetrates the material, through the $e^{-k_{imag}z}$ term. The inverse of k_{imag} has units of meters, and is often called the *skin depth*, δ.

$$\boxed{\delta = \frac{1}{k_{imag}}}$$

Equations for how wavelength and skin depth relate to complex $\tilde{n}$ and complex $\tilde{\epsilon}_r$ can be obtained via these two boxed equations and the "basic relationships" equations above.

The decay of the wave is also sometimes characterized by the *absorption coefficient*, α, which describes the fall off in intensity according to $I = I_0 e^{-\alpha z}$. It's very similar to k_{imag} itself but because intensity varies with amplitude squared there is an added factor of two:

$$\alpha = 2k_{imag}$$

Bad information in Griffiths section 9.4.1

There is some bad information in Griffiths section 9.4.1 (both editions). This handout will strive to correct that information. Let's pick up the story with Eq. 9.124 (both editions), since everything seems OK until that point. Here's that equation:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i\mu\sigma\omega \qquad \text{Eq. 9.124}$$

Griffiths considers the first term to be a positive real number and the second term to be a positive imaginary number, so that $\tilde{k}^2$ is in the first quadrant. He them puts $\tilde{k}^2$ in polar form (maybe not explicitly, but that's what he is doing), takes the square root, and then divides the resulting $\tilde{k}$ into its real and imaginary parts. The resulting equations for k_{real} and k_{imag} are then given as Eq. 9.126 (both editions). Everything seems great, right? Wrong.

The issue here is that in order to put $\tilde{k}^2$ in its polar form the way Griffiths has implicitly done, and get the same result for its square root that Griffiths has gotten in Eq. 9.126, one must assume that the quantities μ, ϵ, and σ are all real. While that is very likely true for μ (because $\mu \approx \mu_0$ for nearly all materials), we have seen/will see several cases in this class where ϵ and σ can be complex! So the square root of $\tilde{k}^2$ is in general NOT given by the formulas Griffiths has calculated, and Eq. 9.126 is not generally true. It's only true for the special case of when ϵ and σ are both real.

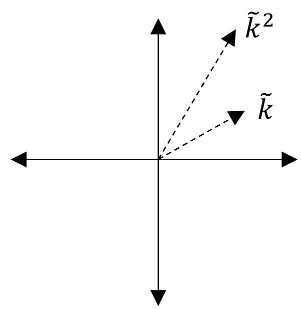
How to fix Eq. 9.126: four cases

So what to do when ϵ and σ are complex? If the values of ϵ and σ are known for a given situation, then you can just numerically compute the square root of $\tilde{k}$ and no analytic solution is needed. However, if you want an algebraic expression for the real and imaginary components of the square root, i.e. an analog of Eq. 9.126, then the situation is best analyzed by dividing it up into some common cases. My inspiration for these four cases comes from the textbook *Foundations of Electromagnetic Theory* by Reitz, Milford, and Christy (4th edition, page 428).

Because $\tilde{k}$ leads directly to $\tilde{n}$ via the equation given above, $\tilde{n} = \frac{c}{\omega}\tilde{k}$, all of the equations below for k_{real} and k_{imag} can be used to obtain the real and imaginary indices of refraction, as well.

<u>Case 1</u>. $\boldsymbol{\epsilon}$ and $\boldsymbol{\sigma}$ are both real

OK, this is the situation that Griffiths assumes. Then $\tilde{k}^2$ is in the first quadrant, and its square root $\tilde{k}$ is also in the first quadrant. When plotted on the complex plane the situation looks like the figure to the right.



The results are what's given in Griffiths Eq. 9.126:

$$\boxed{\begin{aligned} k_{real} &= \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}+1\right]^{1/2} \\ k_{imag} &= \omega\sqrt{\frac{\epsilon\mu}{2}}\left[\sqrt{1+\left(\frac{\sigma}{\epsilon\omega}\right)^2}-1\right]^{1/2} \end{aligned}}$$

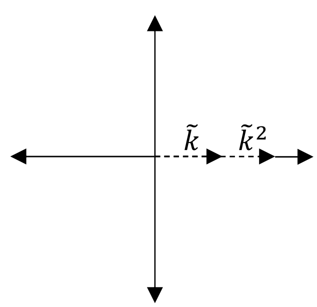
<u>Case 2</u>. A poor conductor: $|\boldsymbol{\sigma}| \ll |\boldsymbol{\epsilon}|\boldsymbol{\omega}$

A good way to define whether a conductor is "good" or "poor" is to compare its magnitude to $|\epsilon|\omega$. For a poor conductor, where $|\sigma| \ll |\epsilon|\omega$, Eq. 9.124 then becomes this:

$$\tilde{k}^2 = \mu\epsilon\omega^2 + i(\text{small real} + \text{small imaginary})$$

$$\tilde{k}^2 \approx \mu\epsilon\omega^2$$

Then $\tilde{k}^2$ is on the x-axis, and its square root $\tilde{k}$ is also on the x-axis (real).



$$\tilde{k} \approx \sqrt{\mu\epsilon\omega^2}$$

That means:

$$\boxed{\begin{aligned} k_{real} &\approx \sqrt{\mu\epsilon\omega^2} \\ k_{imag} &\approx 0 \end{aligned}}$$

In other words, $\tilde{k}$ is nearly purely real; which therefore also means that $\tilde{n}$ is nearly purely real.

Cases 3 and 4. A good conductor: $|\boldsymbol{\sigma}| \gg |\boldsymbol{\epsilon}|\boldsymbol{\omega}$

Having a good conductor means that $|\sigma| \gg |\epsilon|\omega$. Often the conductor will have either a large real part or a large imaginary part at the frequency of interest. This can be seen by the AC conductance of a material which, as derived in a homework problem, is given by:

$$\sigma = \frac{\sigma_0}{1 - i\omega\tau}$$

If $\omega \ll 1/\tau$ then σ will be nearly purely real. If $\omega \gg 1/\tau$ then σ will be nearly purely imaginary. Let's tackle them in the reverse order.

<u>Case 3</u>**: A good conductor, high frequencies ($\boldsymbol{\sigma}$ = nearly purely imaginary)**

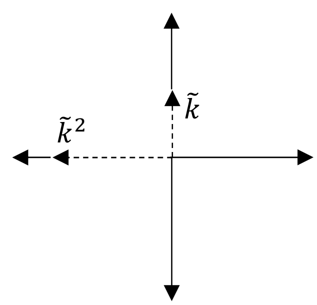
If ω is large, namely $\omega \gg 1/\tau$, then $\sigma \approx \frac{\sigma_0}{-i\omega\tau}$ which is nearly purely imaginary. This is typically true for frequencies in the upper infrared or higher. Eq. 9.124 then becomes this:

$$\tilde{k}^2 = (\text{small real}) + i\mu\sigma\omega$$

Plugging in $\sigma \approx i|\sigma|$, since it's nearly purely imaginary, we have:

$$\tilde{k}^2 \approx -\mu|\sigma|\omega$$

Then $\tilde{k}^2$ is on the negative x-axis, and its square root $\tilde{k}$ is on the y-axis (purely imaginary). Specifically:



$$\tilde{k} \approx i\sqrt{\mu|\sigma|\omega}$$

That means:

$$\boxed{\begin{aligned} k_{real} &\approx 0 \\ k_{imag} &\approx \sqrt{\mu|\sigma|\omega} \end{aligned}}$$

Since $\tilde{k}$ is nearly purely imaginary; $\tilde{n}$ is also nearly purely imaginary.

<u>Case 4</u>**: A good conductor, low frequencies ($\boldsymbol{\sigma}$ = nearly purely real)**

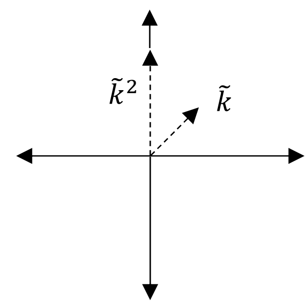
If ω is small, namely $\omega \ll 1/\tau$, then $\sigma \approx \sigma_0$ which is nearly purely real. This is typically true for frequencies which are microwave or lower. Eq. 9.124 then becomes this:

$$\tilde{k}^2 = (\text{small real}) + i\mu\sigma\omega$$

Plugging in $\sigma \approx \sigma_0$, we have:

$$\tilde{k}^2 \approx i\mu\sigma_0\omega$$

Then $\tilde{k}^2$ is on the y-axis, and its square root $\tilde{k}$ is at 45°. Specifically:



$$\tilde{k} \approx \sqrt{\frac{\mu\sigma_0\omega}{2}} + \sqrt{\frac{\mu\sigma_0\omega}{2}}\,i$$

That means:

$$\boxed{\begin{aligned} k_{real} &\approx \sqrt{\frac{\mu\sigma_0\omega}{2}} \\ k_{imag} &\approx \sqrt{\frac{\mu\sigma_0\omega}{2}} \end{aligned}}$$

Since $\tilde{k}$ is a complex number at 45°, $\tilde{n}$ is also a complex number at 45°.