Bessel Functions

by Dr. Colton, Physics 442/471 (last updated: Winter 2024)

General Information

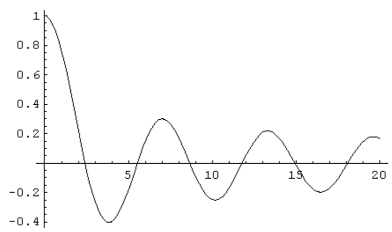
The Bessel functions, $J_\alpha(x)$ are a set of functions for (typically) integer values of α, which:

(a) come up often, especially in the context of differential equations in cylindrical coordinates
(b) have interesting properties
(c) are well understood and have been studied for centuries

They are typically only used for positive values of x. Here are plots of the first four Bessel functions.

BesselJ[α,x] is a built-in *Mathematica* function just like Sin[x], and gives $J_\alpha(x)$

```
In[25]:= Plot[BesselJ[0, x], {x, 0, 20}]
```




```
Out[25]= - Graphics -
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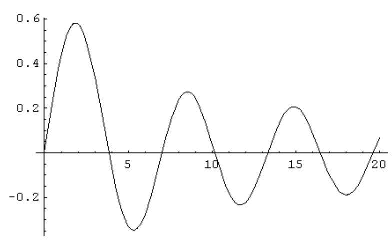
$J_0(x)$
crosses zero at 2.405, 5.520, 8.654, …

← The only one that is not zero at the origin

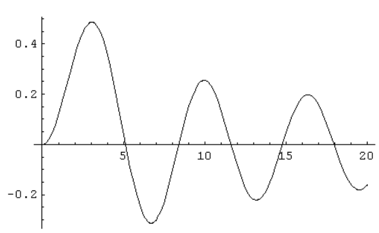
```
In[26]:= Plot[BesselJ[1, x], {x, 0, 20}]
```




```
Out[26]= - Graphics -
```

$J_1(x)$
crosses zero at 3.832, 7.016, 10.173, …

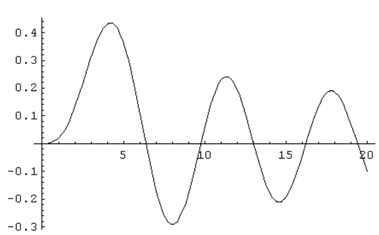
```
In[27]:= Plot[BesselJ[2, x], {x, 0, 20}]
```




```
Out[27]= - Graphics -
```

$J_2(x)$
crosses zero at 5.136, 8.417, 11.620, …

```
In[28]:= Plot[BesselJ[3, x], {x, 0, 20}]
```

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Out[28]= - Graphics -
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$J_3(x)$
crosses zero at 6.380, 9.761, 13.015, …

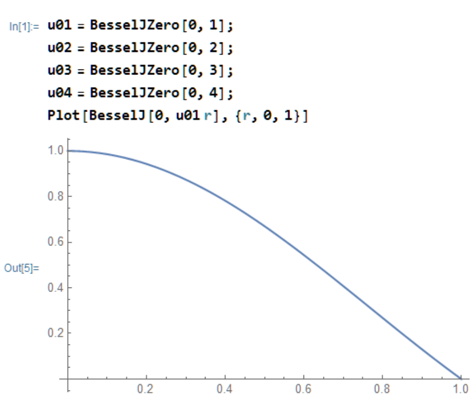
Important facts

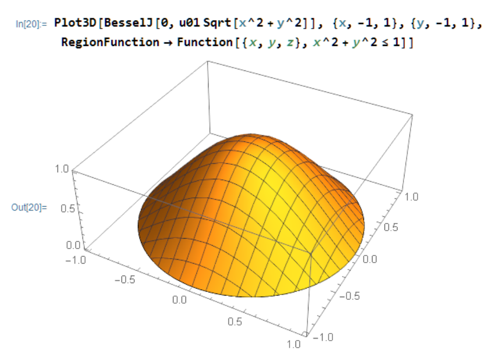
- Bessel's equation is $x^2 \frac{d^2 f}{dx^2} + x\frac{df}{dx} + (x^2 - \alpha^2)f = 0$, has solution $J_\alpha(x)$.
- Bessel functions can be computed via a series formula: $J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\alpha}}{k!(k+\alpha)!2^{2k+\alpha}}$.
- A second set of solutions to Bessel's equation exist, called the "Bessel functions of the second kind". They are written as $Y_\alpha(x)$ or sometimes as $N_\alpha(x)$. They diverge at $x = 0$ and therefore can typically be discounted as viable physical solutions.
  - Various other linear combinations of $J_\alpha(x)$ and $Y_\alpha(x)$ are also solutions to Bessel's equation and are sometimes used; two examples are the "modified Bessel functions" and the "Hankel functions", but they are beyond the scope of this course.
  - The so-called "spherical Bessel functions" and "spherical Hankel functions" are solutions to a different, albeit closely related, differential equation. They are also beyond the scope of this course.
- Derivatives:
  - For $\alpha = 0$: $\frac{d}{dx}\big(J_0(x)\big) = -J_1(x)$ (which means the max/min of J_0 are the zeroes of J_1)
  - For $\alpha \geq 1$: $\frac{d}{dx}\big(J_\alpha(x)\big) = \frac{1}{2}\big(J_{\alpha-1}(x) - J_{\alpha+1}(x)\big)$
- Zeroes: $u_{\alpha m}$ represents the m^th zero of $J_\alpha(x)$. From the previous page we have:
  - $u_{01} = 2.405$, $u_{02} = 5.520$, $u_{03} = 8.654$, …
  - $u_{11} = 3.832$, $u_{12} = 7.016$, $u_{13} = 10.173$, …
  - etc.

  These numbers are available in Mathematica via the BesselJZero function; for example, BesselJZero[0,1] yields a result of 2.4048255576957727686…

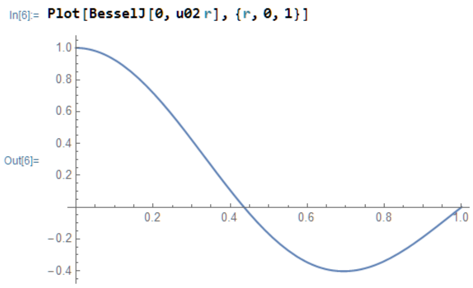
With the substitution: $x = u_{\alpha m} r$

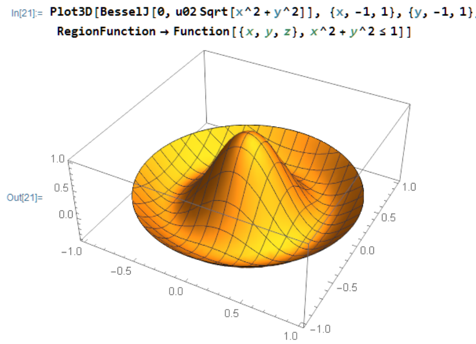
The Bessel functions are often used with the substitution $x = u_{\alpha m} r$, with the domain then restricted to $0 \leq r \leq 1$. The variable r here represents the radial cylindrical coordinate. This then gives rise to a set of functions for each α, labeled by m. Note that whereas the integers α go from 0, 1, 2, 3, etc., the integers m go from 1, 2, 3, 4, etc. Here are the first four functions of the $\alpha = 0$ series, plotted both as 1D functions of r, and as 2D functions with r as the cylindrical coordinate. These are all the $J_0(x)$ Bessel function, just scaled so that more and more of the function gets displayed between 0 and 1.



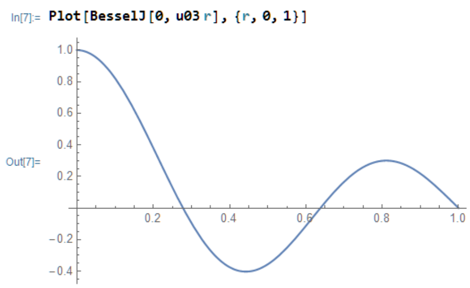


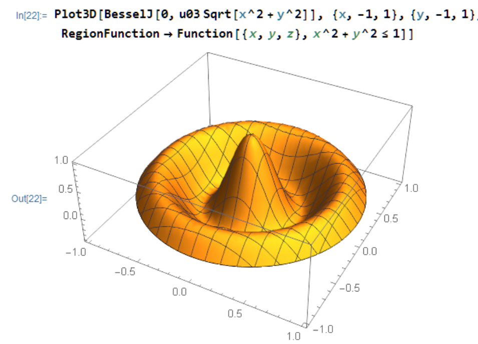
$J_0(u_{01} r)$



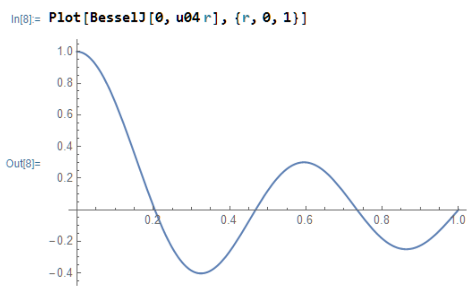


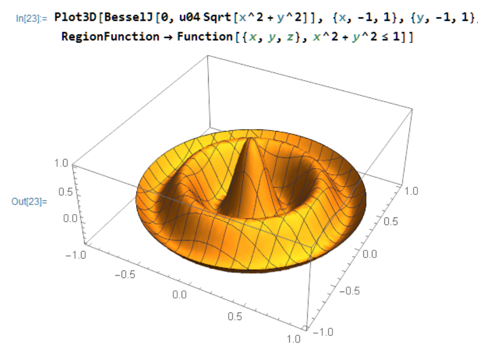
$J_0(u_{02}r)$





$J_0(u_{03}r)$





$J_0(u_{04}r)$

A similar series of plots could be made for $\alpha = 1$, $\alpha = 2$, etc.

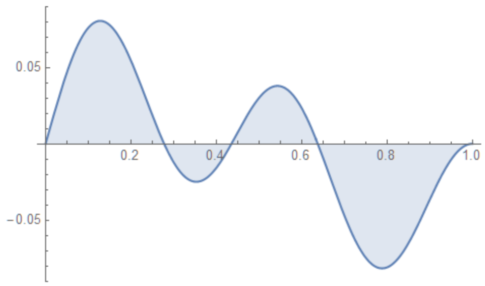
Important facts about the $J_\alpha(u_{\alpha m}r)$ series of functions for a given α

- Bessel's equation becomes $r^2\frac{d^2f}{dr^2} + r\frac{df}{dr} + ({u_{\alpha m}}^2 r^2 - \alpha^2)f = 0$, has solution $J_\alpha(u_{\alpha m}r)$.
- Orthogonality for $\alpha = 0$: $J_0(u_{0m}r)$ and $J_0(u_{0n}r)$ are orthogonal over the interval (0,1) with respect to a weighting function of r:
    - $\int_0^1 J_0(u_{0m}r)J_0(u_{0n}r)r\,dr = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2}\left(J_1(u_{0m})\right)^2 & \text{if } n = m \end{cases}$
- Orthogonality for general α: $J_\alpha(u_{\alpha m}r)$ and $J_\alpha(u_{\alpha n}r))$ are orthogonal over the same interval with the same weighting function:
    - $\int_0^1 J_\alpha(u_{\alpha m}r)J_\alpha(u_{\alpha n}r)r\,dr = \begin{cases} 0 & \text{if } n \neq m \\ \frac{1}{2}\left(J_{\alpha+1}(u_{\alpha m})\right)^2 & \text{if } n = m \end{cases}$

Orthogonality, depicted

I have randomly chosen two functions in the $\alpha = 0$ series (plots given above), namely $m = 2$ and $m = 3$.

```
In[40]:= Plot[BesselJ[0, u02 r] BesselJ[0, u03 r] r, {r, 0, 1}, Filling → Axis]
Out[40]=
```



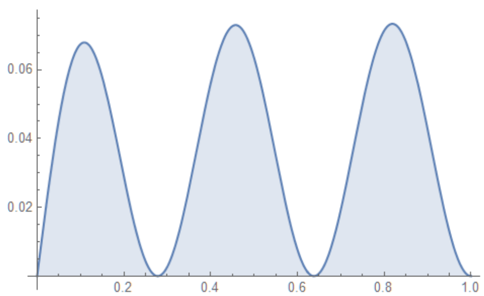
The two functions are orthogonal over the domain of $0 \leq r \leq 1$, when multiplied by a weighting function of r. You can see that the positive areas cancel out the negative areas.

```
In[41]:= Integrate[BesselJ[0, u02 r] BesselJ[0, u03 r] r, {r, 0, 1}]
Out[41]= 0
```

The integral is exactly zero.

$\int_0^1 J_0(u_{0m} r) J_0(u_{0n} r) r dr$
equals 0 if $n \neq m$

```
In[42]:= Plot[BesselJ[0, u03 r] BesselJ[0, u03 r] r, {r, 0, 1}, Filling → Axis]
Out[42]=
```



By contrast, $J_0(u_{03} r)$ is not orthogonal to itself.

```
In[43]:= Integrate[BesselJ[0, u03 r] BesselJ[0, u03 r] r, {r, 0, 1}]
Out[43]= 1/2 BesselJ[1, BesselJZero[0, 3]]^2
```

$\int_0^1 J_0(u_{0m} r) J_0(u_{0n} r) r dr$
equals $\frac{1}{2} \left(J_1(u_{0m}) \right)^2$ if $n = m$

Comparison between Bessel functions and sine/cosine functions

	Sines/Cosines	Bessel functions
1.	Two oscillatory functions: $\sin(x)$ and $\cos(x)$. Often one of them is not used, due to the symmetry of the problem.	Two oscillatory functions for each α: $J_\alpha(x)$ and $Y_\alpha(x)$. Typically Y_α is not used because it's infinite at the origin.
2.	You can determine the value of $\sin(x)$ and $\cos(x)$ for arbitrary x by using a calculator or computer program.	You can determine the value of $J_\alpha(x)$ for arbitrary x by using a calculator or computer program.
3.	Series solutions: $\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$ $\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$	Series solution: $J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\alpha}}{k!(k+\alpha)!2^{2k+\alpha}}$
	Consider just $\sin(x)$:	Consider just $J_\alpha(x)$ for one α, say $\alpha = 0$: (similar things hold true for all α's)
4.	The zeroes of $\sin(x)$ are at $x = \pi, 2\pi, 3\pi$, etc. $x =$ "$m\pi$" is the m^{th} zero	The zeroes of $J_0(x)$ are at $x \approx 2.405, 5.520, 8.654$, etc. $x =$ "u_{0m}" is the m^{th} zero
5.	$\sin(m\pi x)$ has $m-1$ nodes in the interval from 0 to 1. At $x = 1$, $\sin(m\pi x) = 0$ for all m.	$J_0(u_{0m}r)$ has $m-1$ nodes in the interval from 0 to 1. At $r = 1$, $J_0(u_{0m}r) = 0$ for all m.
6.	The differential equation satisfied by $f = \sin(x)$ is $f'' + f = 0$.	The differential equation satisfied by $f = J_0(x)$ is $x^2 f'' + x f' + (x^2 - 0^2) f = 0$.
	The differential equation satisfied by $f = \sin(m\pi x)$ is $f'' + (m\pi)^2 f = 0$.	The differential equation satisfied by $f = J_0(u_{0m}r)$ is $r^2 f'' + r f' + (u_{0m}^2 r^2 - 0^2) f = 0$. $0^2 \rightarrow \alpha^2$ for other α's
7.	$\sin(m\pi x)$ is orthogonal to $\sin(n\pi x)$ on the interval (0,1): $\int_0^1 \sin(m\pi x)\sin(n\pi x)\,dx = \begin{cases} 0, if\ n \neq m \\ \frac{1}{2}, if\ n = m \end{cases}$	$J_0(u_{0m}r)$ is orthogonal to $J_0(u_{0n}r)$ on the interval (0,1), with respect to a weighting function of r: $\int_0^1 J_0(u_{0m}r) J_0(u_{0n}r) r\,dr = \begin{cases} 0, if\ n \neq m \\ \frac{1}{2}\left(J_1(u_{0m})\right)^2, if\ n = m \end{cases}$

Additionally, the Bessel functions are related to sines/cosines through this integral formula:

$$J_\alpha(x) = \frac{1}{\pi}\int_0^\pi \cos(\alpha\theta - x\sin\theta)\,d\theta$$

Quote from Mary Boas, in *Mathematical Methods in the Physical Sciences*: "In fact, if you had first learned about $\sin(nx)$ and $\cos(nx)$ as power series solutions of $y'' = -n^2 y$, instead of in elementary trigonometry, you would not feel that Bessel functions were appreciably more difficult or strange than trigonometric functions. Like sines and cosines, Bessel functions are solutions of a differential equation; they are tabulated and their graphs can be drawn; they can be represented as a series; and a large number of formulas about them are known."