# Derivation of Fresnel Equations for s-polarization

**by Dr. Colton, Physics 471** (last updated: 29 Jan 2024)

using the method of Griffiths, *Introduction to Electrodynamics*

For s-polarization the electric field is perpendicular to the plane of incidence. This is out of the page for this image.

The direction of the magnetic field is chosen the make $\vec{\mathbf{E}} \times \vec{\mathbf{B}}$ be in the direction of propagation.

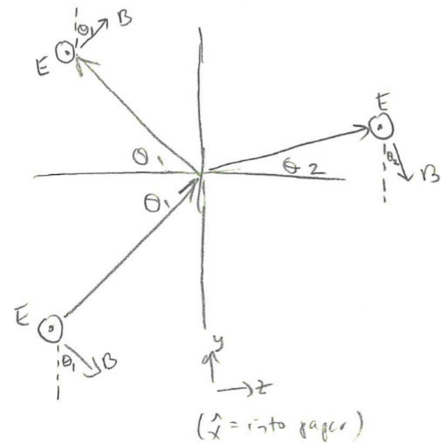


**Figure 1.** Incident, reflected, and transmitted plane wave fields at a material interface.

<u>incident</u>

$$\tilde{\mathbf{E}}_\mathrm{i} = \tilde{E}_{0i} e^{i(\mathbf{k_i}\cdot\mathbf{r}-\omega t)}(-\hat{\mathbf{x}})$$

$$\tilde{\mathbf{B}}_\mathrm{i} = \frac{1}{v_1}\tilde{E}_{0i} e^{i(\mathbf{k_i}\cdot\mathbf{r}-\omega t)}\,(-\cos\theta_1\,\hat{\mathbf{y}} + \sin\theta_1\,\hat{\mathbf{z}})$$

<u>reflected</u>

$$\tilde{\mathbf{E}}_\mathrm{r} = \tilde{E}_{0r} e^{i(\mathbf{k_r}\cdot\mathbf{r}-\omega t)}\,(-\hat{\mathbf{x}})$$

$$\tilde{\mathbf{B}}_\mathrm{r} = \frac{1}{v_1}\tilde{E}_{0r} e^{i(\mathbf{k_r}\cdot\mathbf{r}-\omega t)}\,(+\cos\theta_1\,\hat{\mathbf{y}} + \sin\theta_1\,\hat{\mathbf{z}})$$

<u>transmitted</u>

$$\tilde{\mathbf{E}}_\mathrm{t} = \tilde{E}_{0t} e^{i(\mathbf{k_t}\cdot\mathbf{r}-\omega t)}\,(-\hat{\mathbf{x}})$$

$$\tilde{\mathbf{B}}_\mathrm{t} = \frac{1}{v_2}\tilde{E}_{0t} e^{i(\mathbf{k_t}\cdot\mathbf{r}-\omega t)}\,(-\cos\theta_2\,\hat{\mathbf{y}} + \sin\theta_2\,\hat{\mathbf{z}})$$

<u>Boundary condition 1:</u> $(E_\parallel)_1 = (E_\parallel)_2$ this is the x-component

$$\tilde{E}_{0i} e^{i(\mathbf{k_i}\cdot\mathbf{r}-\omega t)} + \tilde{E}_{0r} e^{i(\mathbf{k_r}\cdot\mathbf{r}-\omega t)} = \tilde{E}_{0t} e^{i(\mathbf{k_t}\cdot\mathbf{r}-\omega t)}$$

The exponentials must all be equal, so cancel them out.

$$\tilde{E}_{0i} + \tilde{E}_{0r} = \tilde{E}_{0t} \qquad (1)$$

<u>Boundary condition 2:</u> $\frac{1}{\mu_1}(B_\parallel)_1 = \frac{1}{\mu_2}(B_\parallel)_2$ this is the y-component

Nonmagnetic, so $\mu_1 = \mu_2 = \mu_0$. Cancel them out.
Cancel exponentials again.

$$\frac{1}{v_1}\tilde{E}_{0i}(-\cos\theta_1) + \frac{1}{v_1}\tilde{E}_{0r}(+\cos\theta_1) = \frac{1}{v_2}\tilde{E}_{0t}(-\cos\theta_2) \qquad (2)$$

Let $\boxed{\alpha = \frac{\cos\theta_2}{\cos\theta_1}}$ $\boxed{\beta = \frac{\nu_1}{\nu_2} \quad \left(= \frac{n_2}{n_1}\right)}$

Multiply Eq (2) by $\frac{\nu_1}{\cos\theta_1}$ on both sides, remove the tildes for simplicity,

$$E_{0i} + E_{0r} = E_{0t} \tag{3}$$

$$-E_{0i} + E_{0r} = -\alpha\beta E_{0t} \tag{4}$$

Use Eq (3) and Eq (4) and solve for the ratio $\frac{E_{0t}}{E_{0i}}$, call it "t",

$$2E_{0i} = (1 + \alpha\beta)E_{0t}$$

$\boxed{t = \frac{2}{1+\alpha\beta}}$ "transmission coefficient" for s, P&W Eq.3.21

Multiply Eq (3) by $\alpha\beta$ then add to Eq (4):

$$\begin{array}{rl} & \alpha\beta E_{0i} + \alpha\beta E_{0r} = \alpha\beta E_{0t} \\ + & -E_{0i} + E_{0r} = -\alpha\beta E_{0t} \\ \hline & (\alpha\beta - 1)E_{0i} + (1 + \alpha\beta)E_{0r} = 0 \end{array}$$

Solve for the ratio $\frac{E_{or}}{E_{0i}}$, call it "r":

$\boxed{r = \frac{1-\alpha\beta}{1+\alpha\beta}}$ "reflection coefficient" for s, P&W Eq.3.21

The two boxed equations for $r$ and $t$ are the "Fresnel equations" for s-polarized light (with $\alpha$ and $\beta$ defined above).