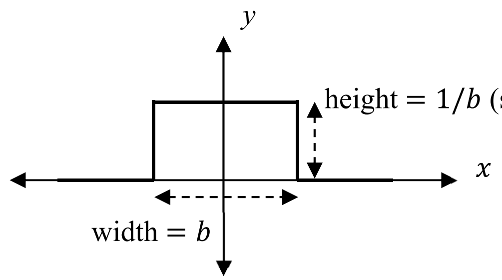
Dirac Delta Functions

by Dr. Colton, Physics 471 (last updated: 11 Jan 2024)

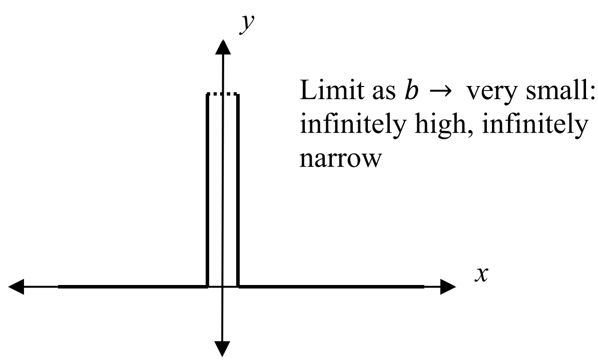
Definitions

1. <u>Definition as limit</u>. The Dirac delta function can be thought of as a rectangular pulse that grows narrower and narrower while simultaneously growing larger and larger.

$\text{rect}(x, b) =$



$\delta(x) =$

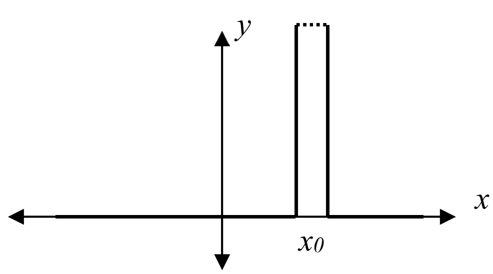


Note that the integral of the delta function is the area under the curve, and has been held constant at 1 throughout the limit process.

$$\int_{-\infty}^{\infty} \delta(x) = 1$$

<u>Shifting the origin</u>. Just as a parabola can be shifted away from the origin by writing $y = (x - x_0)^2$ instead of $y = x^2$, any function can be shifted to the right by writing $x - x_0$ in place of x (here x_0 is positive).

$\delta(x - x_0) =$



Shifting the position of the peak doesn't affect the total area if the integral is taken from $-\infty$ to $+\infty$, or if the interval from a to b contains the peak.

$$\int_{-\infty}^{\infty} \delta(x - x_0) = 1$$

For that matter, since all of the area occurs right in one spot, one can write:

$$\int_a^b \delta(x - x_0) = 1$$

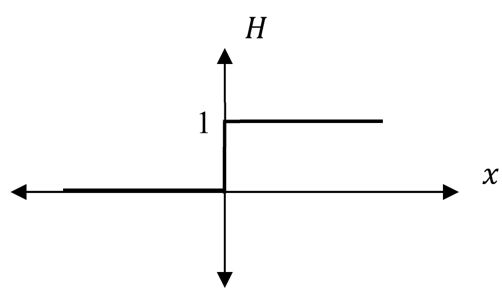
as long as the interval from a to b contains the delta function peak at x_0.

Disclaimer: Mathematicians may object that since the Dirac delta function is only defined in terms of a limit, and/or in terms of how it behaves inside integrals, it is not actually a function but rather a *generalized function* and/or a *functional*. While that is technically true, physicists tend to ignore such issues and recognize that for all practical purposes the delta function can be thought of as a very large and very narrow peak at the origin with an integrated area of 1, as I have described it.

2. Definition as derivative of step function. The step function, also called the "Heaviside step function" is usually defined like this:[1]

$H(x) = 0$ for $x < 0$,
$H(x) = 0.5$ for $x = 0$
$H(x) = 1$ for $x > 0$

It's a function whose only feature is a step up from 0 to 1, at the origin:



Heaviside step function

What's the derivative of this function? Well, the slope is zero for $x < 0$ and the slope is zero for $x > 0$. What about right at the origin? The slope is infinite! So the derivative of this function is a function which is zero everywhere except at the origin, where it's infinite. Also, the integral of that derivative function from $-\infty$ to $+\infty$ will be $H(\infty) - H(-\infty)$, which is 1! An infinitely narrow, infinitely tall peak at the origin whose integral is 1? That's the Dirac delta function! Thus $\delta(x)$ can also be defined as the derivative of the Heaviside step function.

3. Definition as Fourier transform. The Fourier transform of a function gives you the frequency components of the function. What do you get when take the Fourier transform of a pure sine or cosine wave oscillating at ω_0? There is only one frequency component, so the Fourier transform must be a single, very large peak right at ω_0. A delta function![2]

4. Definition as density. What's a function which represents the density of a 1 kg point mass located at the origin? Well, it's a function that must be zero everywhere except at the origin—and it must be infinitely

[1] This is also sometimes called the "theta function" or "Heaviside theta function" and the symbol $\Theta(x)$ is often used for the function (Θ = capital theta).
[2] More technically, $FT\{\cos \omega_0 t\}$ has two real delta function peaks, one at ω_0 and one at $-\omega_0$, and $FT\{\sin \omega_0 t\}$ has two imaginary peaks, an upward one at $-\omega_0$ and a downward one at ω_0.

large at the origin because for a mass that truly occupies only a single point, the mass per volume is infinite. How about its integral? The volume integral of the density must give you the mass, which is 1 kg. A function that is zero everywhere except infinite at the origin, and has an integral equal to 1? It's the Dirac delta function again!

More precisely, that last example would be a three-dimensional analog to the regular delta function $\delta(x)$, because the density must be integrated over three dimensions in order to give the mass. This is typically written $\delta(\mathbf{r})$ or as $\delta^3(\mathbf{r})$, and is the product of three one-dimensional delta functions:

$$\delta^3(\mathbf{r}) = \delta(x)\delta(y)\delta(z)$$

For a delta function located at $\mathbf{r_0} = (x_0, y_0, z_0)$ instead of at the origin, we have:

$$\delta^3(\mathbf{r} - \mathbf{r_0}) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

Properties

1. Integral. One of the most important properties of the delta function has already been mentioned: it integrates to 1.

2. Sifting property. When a delta function $\delta(x - x_0)$ multiplies another function $f(x)$, the product must be zero everywhere except at the location of the infinite peak, x_0. At that location the product is infinite like the delta function, but it must be a "larger" or "smaller" infinity, if those terms make sense to use, depending on whether the value of $f(x)$ at that point is larger or smaller than 1. In other words, the area of the product function is not just 1 anymore, it is 1 times the value of $f(x)$ at the infinite peak $x = x_0$. This is called the "sifting property" of the delta function:

$$\int_{-\infty}^{\infty} \delta(x - x_0) f(x) dx = f(x_0)$$

3. Symmetry. A few other properties can be readily seen from the definition of the delta function:

$\delta(x) = \delta(-x)$	$\delta(x)$ behaves as if it were an even function
$\delta(x - x_0) = \delta\big(-(x - x_0)\big)$	$\delta(x - x_0)$ is symmetric about $x = x_0$

4. Linear systems. If a physical system has linear responses and if its response to delta functions ("impulses") is known, then the output of this system can be determined for almost *any* input, no matter how complex. This rather amazing property of linear systems is a result of the following: almost any arbitrary function can be decomposed into (or "sampled by") a linear combination of delta functions, each weighted appropriately, and each of which produces its own impulse response. Thus, by application of the superposition principle, the overall response to an arbitrary input can be found by adding up all of the impulse responses of the sampled values of the function. This is a complicated paragraph, so give it some thought.

Fourier Transforms

1. FT of a single delta function. The Fourier transform of a single delta function in time can be obtained using the sifting property and the definition of the F.T.:

$$
\begin{aligned}
FT\{\delta(t)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t)e^{i\omega t}dt \\
&= \frac{1}{\sqrt{2\pi}}e^{i\omega(0)} \\
&= \frac{1}{\sqrt{2\pi}}
\end{aligned}
$$

Similarly, the Fourier transform of $\delta(t-t_0)$ is:

$$
\begin{aligned}
FT\{\delta(t-t_0)\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\delta(t-t_0)e^{i\omega t}dt \\
&= \frac{1}{\sqrt{2\pi}}e^{i\omega(t_0)} \\
&= \frac{1}{\sqrt{2\pi}}e^{i\omega t_0}
\end{aligned}
$$

The Fourier transform of $\delta(t-t_0)$, which by definition is a function of ω, has a constant amplitude but a phase that varies linearly with ω.

Another interesting way to define the delta function can be obtained by doing the reverse Fourier transform and setting it equal to the original delta function:

$$
\begin{aligned}
\delta(t-t_0) &= FT^{-1}\left\{\frac{1}{\sqrt{2\pi}}e^{i\omega t_0}\right\} \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{1}{\sqrt{2\pi}}e^{i\omega t_0}\right)e^{-i\omega t}d\omega \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-i\omega(t-t_0)}d\omega \qquad \text{yet another way the Dirac delta function can be defined}
\end{aligned}
$$

Note: the negative sign in the exponent is optional for this definition, as can be seen by substituting $\omega' = -\omega$; the limits get reversed but a negative sign gets added since $d\omega' = -d\omega$, so the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega(t-t_0)}d\omega$ is equal to the integral given in the equation above.

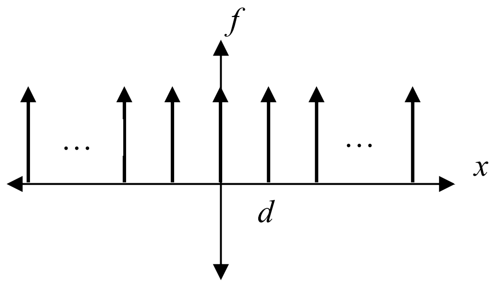
2. <u>FT of two delta functions</u>. As another short example, let's calculate the (spatial) Fourier transform of two delta functions centered on the origin, separated by a distance d. One is at $x = -d/2$ and the other is at $x = +d/2$, therefore

$$
\begin{aligned}
FT\{\text{two deltas}\} &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\delta\left(x+\frac{d}{2}\right)+\delta\left(x-\frac{d}{2}\right)\right)e^{ikx}dx \\
&= \frac{1}{\sqrt{2\pi}}\left(e^{ik\left(-\frac{d}{2}\right)}+e^{ik\left(\frac{d}{2}\right)}\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(2\cos\left(\frac{kd}{2}\right)\right) \\
&= \sqrt{2/\pi}\ \cos\left(\frac{kd}{2}\right)
\end{aligned}
$$

3. FT of a finite Dirac comb. An infinite number of equally spaced delta functions is called a Dirac comb. A *finite* comb function with a delta function at the origin, N delta functions to the left of the origin, and N delta functions to the right of the origin, has $2N + 1$ total delta functions and can be written as:

$$\mathrm{comb}_N(x) = \sum_{j=-N}^{N} \delta(x - x_j)$$

where $x_j = jd$, with d being the spacing between delta functions.



Finite comb function

As is mentioned in the Wikipedia article, https://en.wikipedia.org/wiki/Dirac_comb, one potential use for the comb function is as a sampling function (see "Linear systems," above).

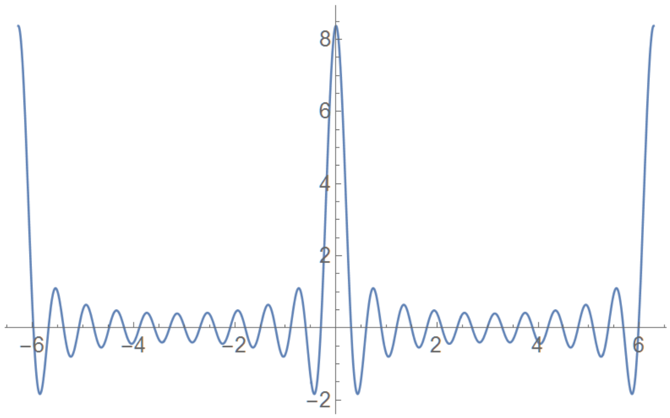
The Fourier transform of the comb function is:

$$\begin{aligned} FT\{\mathrm{comb}_N(x)\} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=-N}^{N} \delta\left(x - x_j\right) e^{ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \sum_{j=-N}^{N} e^{ikx_j} \end{aligned}$$

which, after summing the finite series, doing some algebra, and putting things in terms of the comb spacing d and the total number of delta functions $N_{tot} = 2N + 1$, reduces to:

$$FT\{\mathrm{comb}_N(x)\} = \frac{1}{\sqrt{2\pi}} \frac{\sin\left(\frac{N_{tot} kd}{2}\right)}{\sin\left(\frac{kd}{2}\right)}$$

That is a periodic function made up of large primary peaks surrounded by secondary peaks that decrease in amplitude as you move away from the primary peak.



$FT\{\text{comb}_N(x)\}$ plotted here as a function of k, for $N_{tot} = 21$ and $d = 1$.

Notice that the function of k is periodic and repeats every $2\pi/d$.

As can be found from L'Hospital's Rule, the amplitude of the primary peak depends on N_{tot}:

$$amplitude = \frac{1}{\sqrt{2\pi}} N_{tot}$$

The position of the first zero (a measure of the width of the primary peak) is given by

$$k_{first\ zero} = \frac{2\pi}{N_{tot} d}$$

4. <u>FT of an infinite Dirac comb function</u>: Using the previous two equations, one can see that in the limit as $N_{tot} \to \infty$, the large peaks (like the ones at -2π, 0, and 2π) get infinitely tall and infinitely narrow. They are delta functions! Thus, the Fourier transform of an infinite comb function in x is an infinite comb function in k.