Winter 2024
Physics 471, Final Exam
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barcode here

No time limit. Student calculators are allowed. One page of notes allowed (with printed formulas on one side). Books not allowed.

Name: Solutions

Instructions: Please label & circle/box your answers. **Show your work**, where appropriate. Here are some Stokes vectors & Mueller matrices.

LIGHT	Jones	Stokes
unpolar	n/a	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$
Horiz linear	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$
Vert linear	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$
45° linear	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$
–45° linear	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$
Linear at θ	$\begin{pmatrix}\cos\theta\\ \sin\theta\end{pmatrix}$	$\begin{pmatrix}1\\ \cos 2\theta\\ \sin 2\theta\\ 0\end{pmatrix}$
RCP	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$
LCP	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$
Elliptical	$\begin{pmatrix}A\\ Be^{i\delta}\end{pmatrix}$ with $A^2+B^2=1$	$\frac{1}{E_0^2}\begin{pmatrix}E_0^2\\ E_{0x}^2-E_{0y}^2\\ 2E_{0x}E_{0y}\cos\delta\\ 2E_{0x}E_{0y}\sin\delta\end{pmatrix}$ δ is the phase shift of vert relative to horiz

Angle of elliptical: $\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2AB\cos\delta}{A^2-B^2}\right)$

$E_\alpha = |E_{eff}| \times \sqrt{A^2\cos^2\alpha + B^2\sin^2\alpha + AB\cos\delta\sin 2\alpha}$

$E_{\alpha\pm 90°} = |E_{eff}| \times \sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha - AB\cos\delta\sin 2\alpha}$

Amount polarized: $\sqrt{S_1^2+S_2^2+S_3^2}$

Amount unpolarized: $S_0 - \sqrt{S_1^2+S_2^2+S_3^2}$

Deg of polarization $= \frac{1}{S_0}\sqrt{S_1^2+S_2^2+S_3^2}$

OPTIC	Jones	Mueller (Stokes)
Horiz linear polar	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$
Vert linear polar	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$
45° linear polar	$\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}$
–45° linear polar	$\frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{pmatrix}$
angle θ linear polar	$\begin{pmatrix}\cos^2\theta & \sin\theta\cos\theta\\ \sin\theta\cos\theta & \sin^2\theta\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&\cos 2\theta&\sin 2\theta&0\\ \cos 2\theta&\cos^2 2\theta&\cos 2\theta\sin 2\theta&0\\ \sin 2\theta&\cos 2\theta\sin 2\theta&\sin^2 2\theta&0\\0&0&0&0\end{pmatrix}$
$\lambda/4$ fast axis horiz	$\begin{pmatrix}1&0\\0&i\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$
$\lambda/4$ fast axis vert	$\begin{pmatrix}1&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$
$\lambda/4$ fast axis 45°	$\frac{1}{2}\begin{pmatrix}1+i&1-i\\1-i&1+i\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}$
$\lambda/4$ fast axis angle θ	$\begin{pmatrix}\cos^2\theta+i\sin^2\theta & (1-i)\sin\theta\cos\theta\\ (1-i)\sin\theta\cos\theta & \sin^2\theta+i\cos^2\theta\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&\cos^2 2\theta&\frac{1}{2}\sin 4\theta&-\sin 2\theta\\0&\frac{1}{2}\sin 4\theta&\sin^2 2\theta&\cos 2\theta\\0&\sin 2\theta&-\cos 2\theta&0\end{pmatrix}$
$\lambda/2$ fast axis horiz	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$
$\lambda/2$ fast axis vert	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$
$\lambda/2$ fast axis at θ	$\begin{pmatrix}\cos 2\theta&\sin 2\theta\\ \sin 2\theta&-\cos 2\theta\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&\cos 4\theta&\sin 4\theta&0\\0&\sin 4\theta&-\cos 4\theta&0\\0&0&0&-1\end{pmatrix}$

(12 pts) **Problem 1.** (a) Given two materials with a real index of refraction, prove that the amount of power reflected at an n_1 to n_2 boundary is the same as the amount of power reflected at an n_2 to n_1 boundary. Do NOT assume normal incidence, but you can choose which polarization you'd like to use (it's true for both).

Using p-polarization

$$r = \frac{\alpha - \beta}{\alpha + \beta} \quad \text{where } \alpha = \frac{\cos\theta_2}{\cos\theta_1} \text{ for } 1 \to 2, \quad \frac{\cos\theta_1}{\cos\theta_2} \text{ for } 2 \to 1$$

$$\beta = \frac{n_2}{n_1} \text{ for } 1 \to 2, \quad \frac{n_1}{n_2} \text{ for } 1 \to 2$$

$$r_{12} = \frac{\frac{\cos\theta_2}{\cos\theta_1} - \frac{n_2}{n_1}}{\frac{\cos\theta_2}{\cos\theta_1} + \frac{n_2}{n_1}} \times \frac{n_1\cos\theta_1}{n_1\cos\theta_1} \qquad r_{21} = \frac{\frac{\cos\theta_1}{\cos\theta_2} - \frac{n_1}{n_2}}{\frac{\cos\theta_1}{\cos\theta_2} + \frac{n_1}{n_2}} \times \frac{n_2\cos\theta_2}{n_2\cos\theta_2}$$

$$r_{12} = \frac{n_1\cos\theta_2 - n_2\cos\theta_1}{n_1\cos\theta_2 + n_2\cos\theta_1} \qquad r_{21} = \frac{n_2\cos\theta_1 - n_1\cos\theta_2}{n_2\cos\theta_1 + n_1\cos\theta_2}$$

$$r_{21} = -r_{12}$$

so $|r_{21}|^2 = |r_{12}|^2$

$$R_{21} = R_{12} \quad \checkmark$$

(14 pts) **Problem 2**. A Fabry-Perot interferometer is constructed using two identical partially reflecting mirrors with air in between them, with the light coming in at normal incidence.

(a) Show that $F = 4R/(1-R)^2$, where R is the reflectance of the mirrors.

$$F = \frac{4|r_{10}||r_{12}|}{(1-|r_{10}||r_{12}|)^2} = \frac{4|r_{10}|^2}{(1-|r_{10}|^2)^2} = \frac{4R}{(1-R)^2} \quad \text{since } R = |r|^2$$

since $r_{12} = r_{10}$ here

(b) A krypton lamp emits light near $\lambda = 587$ nm. If the lamp is placed in a strong magnetic field, that spectral line splits into two closely spaced wavelengths due to the Zeeman effect. Determine the parameters R and d for a Fabry-Perot interferometer that would allow you to measure two lines which are separated in wavelength by 0.002 nm, specifically with a free spectral range which is 10× the wavelength separation and a resolution that is 10× better than the $\Delta\lambda_{FWHM}$ needed to resolve them. *Hint*: solving $F = 4R/(1-R)^2$ for R yields $R = \frac{2+F-2\sqrt{1+F}}{F}$, where I've used the quadratic equation and picked the root that is less than 1.

$\Delta\lambda_{FSR} = .02$ nm

$\Delta\lambda_{FWHM} = .0002$ nm

$$\Delta\lambda_{FSR} = \frac{\lambda^2}{2n_1 d\cos\theta_1}$$

$$= \frac{\lambda^2}{2d} \quad \text{since } n_1 = 1,\ \theta_1 = 0^\circ$$

$$d = \frac{\lambda^2}{2\Delta\lambda_{FSR}}$$

$$= \frac{(587 \cdot 10^{-9})^2}{2(.02 \cdot 10^{-9})}$$

$$= .0086\,m$$

$$\boxed{d = 8.6\,mm}$$

$$\frac{\Delta\lambda_{FSR}}{\Delta\lambda_{FWHM}} = \frac{\pi\sqrt{F}}{2}$$

$$F = \left(\frac{2}{\pi}\frac{\Delta\lambda_{FSR}}{\Delta\lambda_{FWHM}}\right)^2$$

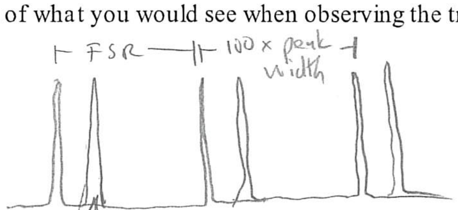
$$= \left(\frac{2}{\pi}\cdot 100\right)^2 = \underline{4052.8}$$

Then $R = \dfrac{2+F-2\sqrt{1+F}}{F}$

$$= \frac{4054.8 - 2\sqrt{4053.8}}{4052.8}$$

$$\boxed{R = 96.9\%}$$

(c) Make a sketch of what you would see when observing the transmission as a function of d for this situation.



peaks from second wavelength are $\frac{1}{10}$ of the FSR

~~(16 pts) **Problem 3**. Second harmonic generation (SHG) is the conversion of light with wavelength λ into light with wavelength $\lambda/2$ (i.e. doubling the frequency), typically achieved by shining intense laser light into a crystal. To do~~

this, both the wavelengths λ and $\lambda/2$ need to travel at the same speed in the material. Unfortunately, the index of refraction is almost never the same for different wavelengths in a material (due to dispersion). However, because there is a difference in index of refraction for the two different polarizations in uniaxial and biaxial crystals, it is sometimes possible to make the index of refraction for one polarization for wavelength λ equal to the index of refraction for the other polarization for wavelength $\lambda/2$. This is called "phase matching" because when traveling at the same speed, the two wavelengths stay in phase. Specifically, in "Type I SHG", two photons of wavelength λ with the ordinary index of refraction will combine to form a single photon of wavelength $\lambda/2$ which has the other index. Achieving this requires careful tuning of the angle of the crystal.

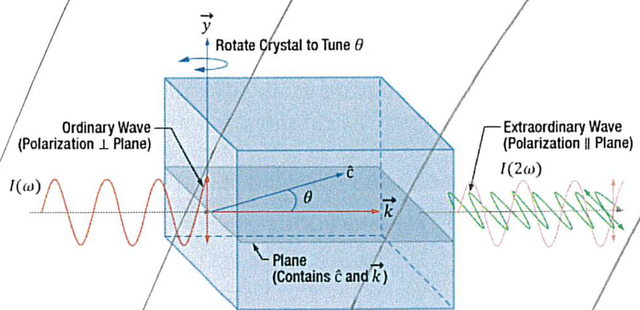


Image from Thorlabs website, https://www.thorlabs.com/newgrouppage9.cfm?objectgroup_id=15444, "SHG Tutorial" tab.

Here "c" is the optic axis, "k" is the laser direction (normal to the surface), and θ is the angle between the two.

In my lab I have an 800 nm "Ti:sapphire" laser which we regularly frequency double using KDP (potassium dihydrogen phosphate) or a closely related uniaxial crystal. At 800 nm KDP has indices $n_o = 1.5015$ and $n_e = 1.4633$. At 400 nm (the second harmonic), the indices are $n_o = 1.5240$ and $n_e = 1.4798$.

What angle θ will make the index of refraction the same for the two wavelengths, and what will that index be?

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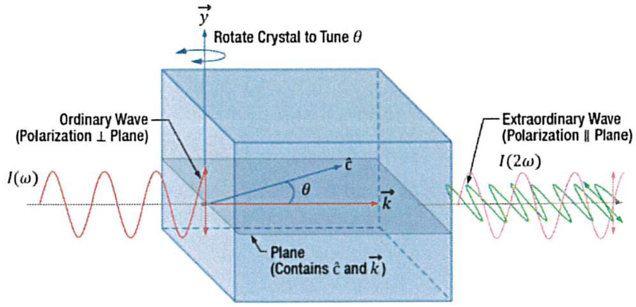

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$800\text{ nm} \Rightarrow n = n_o = \boxed{1.5015}$ this is the index we must match

$$400\text{ nm} \Rightarrow n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta_2 + n_e^2 \cos^2\theta_2}}$$ set equal to 1.5015, solve for θ_2

$$\sqrt{n_o^2 \sin^2\theta_2 + n_e^2\cos^2\theta_2} = \frac{n_o n_e}{n}$$

$$n_o^2 \sin^2\theta_2 + n_e^2(1-\sin^2\theta_2) = \left(\frac{n_o n_e}{n}\right)^2$$

$$(n_o^2 - n_e^2)\sin^2\theta_2 + \cancel{n_e^2} = \frac{n_o^2 n_e^2}{n^2} - n_e^2$$

$$\sin\theta_2 = \sqrt{\frac{\frac{n_o^2 n_e^2}{n^2} - n_e^2}{n_o^2 - n_e^2}}$$ where n_o, n_e are the 400nm values

$$\sin\theta_2 = \sqrt{\frac{\frac{1.524^2\, 1.4798^2}{1.5015^2} - 1.4798^2}{1.524^2 - 1.4798^2}}$$ plug into calculator, take inverse sine

$$\boxed{\theta_2 = 44.89°}$$

Let's orient the x-axis with the initial polarization $\leftrightarrow$ $\oslash$ 30 ?

(14 pts) **Problem 4.** Malus's law, named after Étienne-Louis Malus (1775-1812), says that when a perfect polarizer is placed in a polarized beam of light, the intensity of light which passes through the polarizer is $I = I_0 \cos^2\theta$, where I_0 is the intensity of the light before the polarizer and θ is the angle between the light's initial polarization and the polarizing axis of the polarizer.

(a) Use Jones or Stokes vectors and associated matrices to derive this law. (Then you are welcome to use this law to answer the rest of the questions).

Jones:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix}_{final} = \begin{pmatrix} \cos^2\theta & \sin\theta\cos\theta \\ \sin\theta\cos\theta & \sin^2\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

normalized

$$= \begin{pmatrix} \cos^2\theta \\ \sin\theta\cos\theta \end{pmatrix}$$

$$I \sim (\cos^2\theta)^2 + (\sin\theta\cos\theta)^2$$

$$= \cos^2\theta\,\underbrace{(\cos^2\theta + \sin^2\theta)}_{1} = \boxed{\cos^2\theta}$$

Stokes:

$$\begin{pmatrix} S_0 \\ S_1 \\ S_2 \\ S_3 \end{pmatrix}_{Final} = \frac{1}{2}\begin{pmatrix} 1 & \cos 2\theta & \sin 2\theta & 0 \\ \cos 2\theta & \cos^2 2\theta & \cos 2\theta \sin 2\theta & 0 \\ \sin 2\theta & \cos 2\theta \sin 2\theta & \sin^2 2\theta & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$= \frac{1}{2}\begin{pmatrix} 1 + \cos 2\theta \\ \cos 2\theta + \cos^2 2\theta \\ \sin 2\theta + \cos 2\theta \sin 2\theta \\ 0 \end{pmatrix}$$

So gives intensity

$$= \frac{1}{2}(1 + \cos 2\theta)$$

$2\cos^2\theta - 1$

$$= \boxed{\cos^2\theta}$$

(b) If a beam polarized at 0° is sent into a polarizer at 90°, no light emerges. However, if a polarizer at 45° is placed between the two, some light does. What fraction of the original intensity is that?

$\leftrightarrow$ $\oslash$ Φ

$\cos^2 45°$ $\cos^2 45°$

$$\frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = \boxed{25\%}$$

(c) What fraction of the original intensity gets through if you have three polarizers, placed at 30°, 60°, and then 90°?

$$\leftrightarrow \;\oslash\; \oslash\; \Phi \rightarrow (\cos^2 30°)(\cos^2 30°)(\cos^2 30°)$$

$$= \cos^6 30°$$

$$= \boxed{42.19\%}$$

(d) What fraction gets through if you have N polarizers, placed at angles of $90°/N$, $2 \times 90°/N$, $3 \times 90°/N$, etc.

$$\left(\cos^2\frac{90°}{N}\right) \times \left(\cos^2\frac{90}{N}\right) \times \ldots \;(N \text{ times}) = \left(\cos^2\frac{90°}{N}\right)^N$$

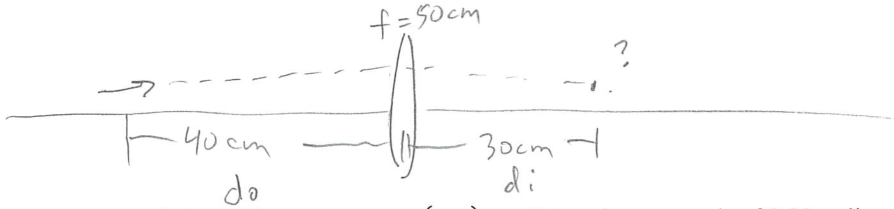
$$= \boxed{\left[\cos\left(\frac{90°}{N}\right)\right]^{2N}} \quad \text{or} \quad \left[\cos\left(\frac{\pi}{2N}\right)\right]^{2N} \text{ if you like radians}$$

(e) Determine the limit as $N \to$ infinity. I don't need an official proof; you can just answer this by trying increasing values for N in your answer to the previous part until the limit is clear.

$N = 2 \to 25\%$
$3 \to 42.2\%$
$5 \to 60.5\%$
$10 \to 78.1\%$

$N = 100 \to 97.6\%$
$N = 1000 \to 99.8\%$
$N = 10000 \to 99.98\%$

Converges to 100% !



(16 pts) **Problem 5**. A light ray leaves the point $(x, y) = (0, 1\text{ cm})$ at an angle of 0.02 radians (which is 1.15°). It travels 40 cm along the x-axis, then runs into a thin lens having focal length $f = 50$ cm. After another 30 cm, what will be the light ray's vertical position and angle?

Can use the "d_o, then optic, then d_i" matrix, where optic $= \begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$ (thin lens)
from formula sheet

$$\begin{pmatrix} h \\ \theta \end{pmatrix}_{final} = \begin{pmatrix} A + d_i C & d_o A + B + d_o d_i C + d_i D \\ C & d_o C + D \end{pmatrix} \begin{pmatrix} h \\ \theta \end{pmatrix}_{initial}$$

$$= \begin{pmatrix} 1 + d_i(-\frac{1}{f}) & d_o(1) + d_o d_i(-\frac{1}{f}) + d_i(1) \\ -\frac{1}{f} & d_o(-\frac{1}{f}) + 1 \end{pmatrix} \begin{pmatrix} h \\ \theta \end{pmatrix}$$

$$= \begin{pmatrix} .4 & 46 \\ -.02 & +.2 \end{pmatrix} \begin{pmatrix} h \\ \theta \end{pmatrix}$$

New vertical position $h_f = .4h + 46\theta$

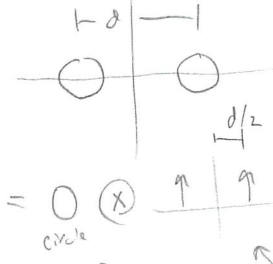
$= .4(1) + 46(.02) = \boxed{1.32\text{ cm}}$

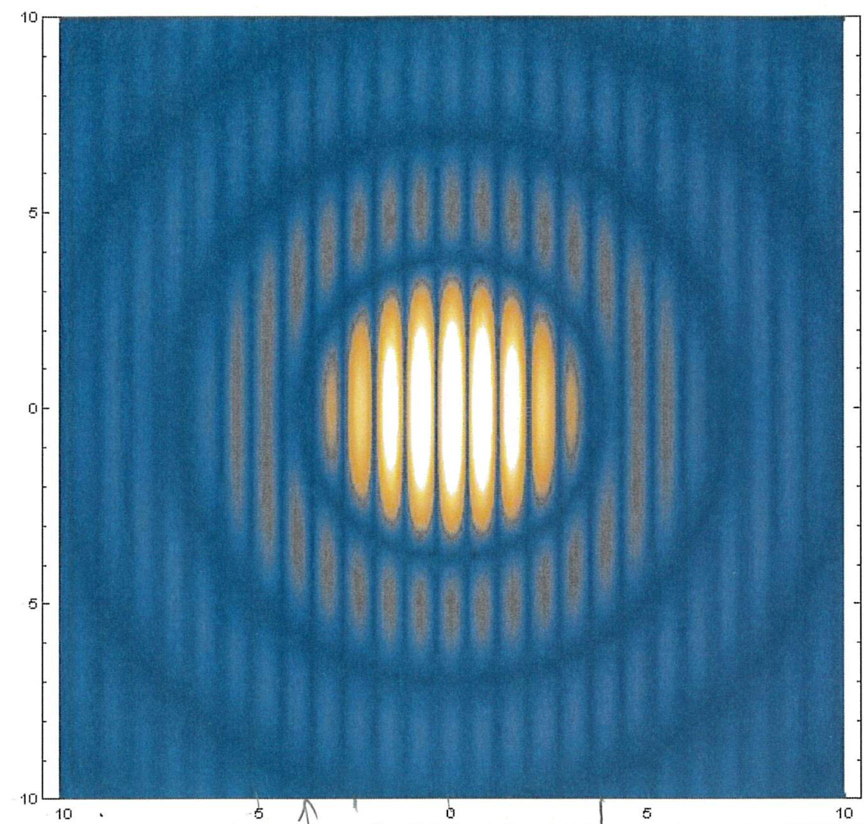
New angle $\theta_f = -.2h + .2\theta$

$= -.02(1) + .2(.02) = \boxed{-.016\text{ rad}}$

$= -.917°$

(14 pts) **Problem 6.** The following is the observed diffraction pattern on a screen placed 5 m away from <u>two circular apertures</u> each having radius a, with their separation distance being d. The laser is orange, with $\lambda = 585$ nm. Deduce the aperture radius a from the given experimental data. The scale bars indicate cm, so the printed image is about 50% smaller than "real life". *Hint*: the first zero of the jinc(x) function occurs at $x = 1.22\pi$.





$$FT = FT(\bigcirc)\cdot FT(\uparrow\uparrow)$$

$$= \frac{a^2}{2}\,\mathrm{jinc}(k_\rho a)\underbrace{\left(e^{\frac{ik_x d}{2}} + e^{-\frac{ik_x d}{2}}\right)}_{2\cos\frac{k_x d}{2}}$$

$$I \sim FT^2$$

$$I = I_0\,\underline{\mathrm{jinc}^2\left(\frac{k\rho a}{z}\right)}\cos^2\left(\frac{k x d}{2z}\right)$$

forms first zero when $\frac{k\rho a}{z} = 1.22\pi$

$k = \frac{2\pi}{\lambda}$

first zero looks about $\rho = 3.725$
(halfway from 2.5 to 5)

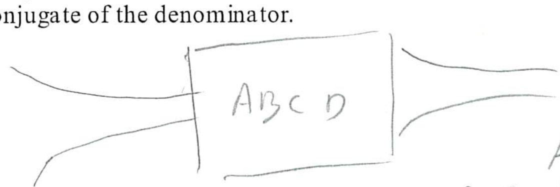
$$\frac{2\pi}{\lambda}\frac{\rho_{zero}\, a}{z} = 1.22\pi$$

$$a = \frac{1.22\,\lambda\, z}{2\cdot\rho_{zero}} = \frac{1.22\cdot(585\cdot 10^{-9})(5)}{2\,(3.725\cdot 10^{-2})}$$

$$a = 4.79\cdot 10^{-5}\ \mathrm{m}$$

$$\boxed{a = 47.9\ \mu\mathrm{m}}$$

(14 pts) **Problem 7.** A laser beam with wavelength λ is focused to a beam waist of width w_0, and enters a complex optical system at its focus. The optical system is described by a given ABCD matrix. Use the ABCD law for Gaussian beams to determine the q-parameter and hence the nature of the beam coming out of the system. Specifically: in terms of λ, w_0, A, B, C, and D, what will be the width of the beam waist of the new beam, and where will it occur relative to the end of the system described by that ABCD matrix? Hint: to separate a fraction that has a complex denominator into its real and imaginary parts, multiply numerator and denominator by the complex conjugate of the denominator.



$$q_f = \frac{Aq_i + B}{Cq_i + D} = \underbrace{z_f + i z_{0,f}}_{\text{separate } q_f \text{ into real + imaginary}}$$

separate q_f into real + imaginary, gives z and z_0 final

$q_i = z_i + i z_{0,i}$

$z = 0$ at focus

$q_i = i z_{0i}$

$$q_f = \frac{A(iz_{0i}) + B}{C(iz_{0i}) + D} \times \frac{D - iCz_{0i}}{D - iCz_{0i}}$$

$$= \frac{BD + ACz_{0i}^2 + i(ADz_{0i} - BCz_{0i})}{D^2 + C^2 z_{0i}^2}$$

real   imag $(AD - BC)z_{0i}$

$$z_{0i} = \frac{kw_0^2}{2} = \frac{2\pi w_0^2}{\lambda \cdot 2}$$

$$q_{f\,real} = \boxed{z_f = \frac{BD + AC\frac{\pi^2 w_0^4}{\lambda^2}}{D^2 + C^2\frac{\pi^2 w_0^4}{\lambda^2}}}$$

$$q_{f\,imag} = z_{0f} = \frac{\pi}{\lambda} w_{0f}^2$$

$$\frac{\pi}{\lambda} w_{0f}^2 = \frac{(AD - BC)\frac{\pi}{\lambda} w_0^2}{D^2 + C^2\frac{\pi^2 w_0^4}{\lambda^2}}$$

$$w_{0f} = \sqrt{\frac{\lambda}{\pi}\frac{(AD-BC)\frac{\pi}{\lambda} w_0^2}{D^2 + C^2\frac{\pi^2 w_0^4}{\lambda^2}}}$$

$$\boxed{w_{0f} = \left(\sqrt{\frac{AD - BC}{D^2 + C^2\frac{\pi^2 w_0^4}{\lambda^2}}}\right) w_0}$$