Physics 471 Formula Sheet (28 Feb 2025)

$c = 2.998 \times 10^8$ m/s $\quad h = 6.626 \times 10^{-34}$ J·s

$k_B = 1.381 \times 10^{-23}$ J/K $\quad N_A = 6.022 \times 10^{23}$

$m_{elec} = 9.109 \times 10^{-31}$ kg $\quad e = 1.602 \times 10^{-19}$ C

$\varepsilon_0 = 8.854 \times 10^{-12}$ C²/N·m²

$\mu_0 = 4\pi \times 10^{-7}$ T·m/A $\quad \sigma = 5.670 \times 10^{-8}$ W/m²·K⁴

<u>General E&M</u>

$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$

$\nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t$

$\nabla \cdot \mathbf{B} = 0$

$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \varepsilon_0 \partial \mathbf{E}/\partial t$

$\nabla \cdot \mathbf{D} = \rho_{free}$

$\nabla \times \mathbf{H} = \mathbf{J}_{free} + \frac{\partial \mathbf{D}}{\partial t}$

$\nabla \cdot \mathbf{J} = -\frac{\partial \rho}{\partial t}$

$\mathbf{D} = \varepsilon_0 \mathbf{E} + \mathbf{P}, \quad = \varepsilon_0 \varepsilon_r \mathbf{E}$ for linear isotropic

$\mathbf{H} = \frac{\mathbf{B}}{\mu_0} - \mathbf{M}, \quad = \frac{\mathbf{B}}{\mu_0 \mu_r}$ for linear isotropic

$\mathbf{P} = \varepsilon_0 \chi \mathbf{E}$ for linear isotropic

$\varepsilon_r = 1 + \chi$ for linear isotropic

$\rho = \rho_{free} + \rho_{bound} \qquad \rho_{bound} = -\nabla \cdot \mathbf{P}$

$\mathbf{J} = \mathbf{J}_{free} + \mathbf{J}_{bound} + \mathbf{J}_{polar}$

$\mathbf{J}_{bound} = \nabla \times \mathbf{M} \qquad \mathbf{J}_{polar} = \partial \mathbf{P}/dt$

$$\nabla^2 E - \varepsilon_0 \mu_0 \frac{\partial^2 \mathbf{E}}{\partial t^2} = \mu_0 \frac{\partial \mathbf{J}_{free}}{\partial t} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} - \frac{1}{\varepsilon_0} \nabla(\nabla \cdot \mathbf{P})$$

Plane wave: $\tilde{\mathbf{E}} = \tilde{\mathbf{E}}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}$

$\tilde{\varepsilon}_r = \tilde{n}^2 \quad \frac{\omega}{k} = \frac{c}{n} \quad \tilde{k} = \frac{\omega}{c}\tilde{n} \quad \lambda = \frac{2\pi}{k_{real}} \quad \delta = \frac{1}{k_{imag}}$

<u>Lorentz Model</u>

$\omega_p^2 = \frac{Nq^2}{m\epsilon_0}$

$\chi = \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\omega\gamma}$ (dielectrics)

$\chi = \frac{\omega_p^2}{-\omega^2 - i\omega\gamma}$ (metals)

<u>Poynting</u>

$\nabla \cdot \mathbf{S} + \frac{\partial u_{field}}{\partial t} = -\frac{\partial u_{medium}}{\partial t}$

$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E} \times \mathbf{B})$

$u_{field} = \frac{\varepsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2$

$\frac{\partial u_{medium}}{\partial t} = \mathbf{E} \cdot \mathbf{J}$

$I(t) = \langle S(t) \rangle = \frac{1}{2} n \varepsilon_0 c E(t)^2$

<u>Fresnel Eqs</u>

$\alpha = \frac{\cos\theta_2}{\cos\theta_1}, \ \beta = \frac{n_2}{n_1}$

$r = \frac{\alpha - \beta}{\alpha + \beta}, \quad t = \frac{2}{\alpha + \beta}$ (p-polar)

$r = \frac{1 - \alpha\beta}{1 + \alpha\beta}, \quad t = \frac{2}{1 + \alpha\beta}$ (s-polar)

$R = |r|^2, \quad T = \alpha\beta |t|^2$

<u>Two Interfaces</u>

$t_{tot} = \frac{t_{01} t_{12}}{\exp(-ik_1 d \cos\theta_1) - r_{10} r_{12} \exp(ik_1 d \cos\theta_1)}$

$T_{tot} = \frac{n_2}{n_0} \frac{\cos\theta_2}{\cos\theta_1} \frac{|t_{01}|^2 |t_{12}|^2}{|1 - r_{10} r_{12} \exp(i2k_1 d \cos\theta_1)|^2}$

$T_{tot} = \alpha_{02} \beta_{02} |t_{02}|^2 = \frac{T_{max}}{1 + F \sin^2 \Phi/2}$

$T_{max} = \frac{T_{01} T_{12}}{(1 - \sqrt{R_{10}} \sqrt{R_{12}})^2}$

$F = \frac{4|r_{10}||r_{12}|}{(1 - |r_{10}||r_{12}|)^2}$

$\Phi = \phi_{10} + \phi_{12} + 2k_1 d \cos\theta_1$

$\Delta\Phi_{\mathrm{FWHM}} = 4/\sqrt{F}$

$\Delta\lambda_{FWHM} = \frac{\lambda^2}{\pi n_1 d \cos\theta_1 \sqrt{F}}$

$\Delta\lambda_{FSR} = \frac{\lambda^2}{2 n_1 d \cos\theta_1}$

$f = 2\pi/\Delta\Phi_{\mathrm{FWHM}} = \Delta\lambda_{FSR}/\Delta\lambda_{FWHM} = \pi\sqrt{F}/2$

<u>Multilayers</u>

$t_{tot} = 1/a_{11}$

$r_{tot} = a_{21}/a_{11}$

$\beta_j = k_j \ell_j \cos\theta_j$

p-polar:

$$M_j = \begin{pmatrix} \cos\beta_j & -\frac{i \sin\beta_j \cos\theta_j}{n_j} \\ -\frac{i n_j \sin\beta_j}{\cos\theta_j} & \cos\beta_j \end{pmatrix}$$

$$A = \frac{1}{2n_0 \cos\theta_0} \begin{pmatrix} n_0 & \cos\theta_0 \\ n_0 & -\cos\theta_0 \end{pmatrix} \times \left(\prod_{j=1}^N M_j\right) \begin{pmatrix} \cos\theta_{N+1} & 0 \\ n_{N+1} & 0 \end{pmatrix}$$

s-polar:

$$M_j = \begin{pmatrix} \cos\beta_j & -\frac{i \sin\beta_j}{n_j \cos\theta_j} \\ -i n_j \sin\beta_j \cos\theta_j & \cos\beta_j \end{pmatrix}$$

$$A = \frac{1}{2n_0 \cos\theta_0} \begin{pmatrix} n_0 \cos\theta_0 & 1 \\ n_0 \cos\theta_0 & -1 \end{pmatrix} \times \left(\prod_{j=1}^N M_j\right) \begin{pmatrix} 1 & 0 \\ n_{N+1} \cos\theta_{N+1} & 0 \end{pmatrix}$$

<u>Crystals</u>

$\frac{1}{n^2} = \frac{u_x^2}{n^2 - n_x^2} + \frac{u_y^2}{n^2 - n_y^2} + \frac{u_z^2}{n^2 - n_z^2}$

Biaxial:

$\cos\theta = \pm \frac{n_x}{n_y} \sqrt{\frac{n_z^2 - n_y^2}{n_z^2 - n_x^2}}$ (optic axes dirs.)

Uniaxial:

$n = n_o, \ n = \frac{n_o n_e}{\sqrt{n_o^2 \sin^2\theta_2 + n_e^2 \cos^2\theta_2}}$

p-polar, optic axis ⊥ to surface:

$\tan\theta_2 = \frac{n_e}{n_o} \frac{\sin\theta_1}{\sqrt{n_e^2 - \sin^2\theta_1}}$

$\tan\theta_s = \frac{n_o}{n_e} \frac{\sin\theta_1}{\sqrt{n_e^2 - \sin^2\theta_1}}$

<u>Polarization</u>

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<u>Fourier, Delta, Convolution</u>

$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$

$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\omega) e^{-i\omega t} d\omega$

$I(\omega) = \frac{1}{2} n \varepsilon_0 c |E(\omega)|^2$

$\int_{-\infty}^{\infty} I(t) dt = \int_{-\infty}^{\infty} I(\omega) d\omega$

$\delta(t - t_0) \Leftrightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t - t_0)} d\omega$

$a(t) \otimes b(t) = \int_{-\infty}^{\infty} a(t') b(t - t') dt'$

$FT\{a(t) \otimes b(t)\} = \sqrt{2\pi}\, FT\{a(t)\} \cdot FT\{b(t)\}$

$FT\{a(t) \cdot b(t)\} = \frac{1}{\sqrt{2\pi}} FT\{a(t)\} \otimes FT\{b(t)\}$

Two deltas: $FT = \sqrt{2/\pi}\ \cos(kd/2)$

Comb function (odd N deltas): $FT = \frac{1}{\sqrt{2\pi}} \frac{\sin Nkd/2}{\sin kd/2}$

$FT\{E_0 e^{-t^2/(2T^2)} e^{-i\omega_0 t}\} = E_0 T e^{-T^2(\omega - \omega_0)^2/2}$

<u>Linear dispersion</u>

$v_g \approx \left(\frac{d}{d\omega} k_{real} \Big|_{\omega = \omega_0} \right)^{-1}$

$t' \approx \frac{d}{d\omega} \mathbf{k}_{\mathbf{real}} \Big|_{\omega = \omega_0} \cdot \Delta\mathbf{r}$

$I \sim e^{-2\mathbf{k}_{\mathbf{imag}}(\omega_0) \cdot \Delta\mathbf{r}} |E(t - t', \mathbf{r_0})|^2$

<u>Quadratic dispersion</u>

$k = k_0 + \frac{1}{v_g}(\omega - \omega_0) + \alpha(\omega - \omega_0)^2$

$\frac{1}{v_g} = \frac{1}{c}(n'\omega + n)|_{\omega = \omega_0}$

$\alpha = \frac{1}{2c}(n''\omega + 2n')|_{\omega = \omega_0}$

Gaussian wavepacket, through thickness z:

$\Phi = 2\alpha z/T^2$

$\tilde{T} = T\sqrt{1 + \Phi^2}$

$$E(t, z) = \frac{E_0 e^{i(kz - \omega_0 t)}}{(1 + \Phi^2)^{1/4}} \exp\left(\frac{i}{2} \tan^{-1}\Phi - \frac{i}{2} \frac{\Phi}{\tilde{T}^2} \left(t - \frac{z}{v_g}\right)^2 \right) \exp\left(-\frac{1}{2\tilde{T}^2} \left(t - \frac{z}{v_g}\right)^2 \right)$$

<u>Michelson, Temporal Coherence</u>

Single ω: $I_{det}(\tau) = 2I_0(1 + \cos\omega\tau)$

Band of ω's:

$\varepsilon = \int_{-\infty}^{\infty} I(\omega) d\omega$

$\gamma(\tau) = \frac{1}{\varepsilon} \int_{-\infty}^{\infty} I(\omega) e^{-i\omega\tau} d\omega$

$\mathrm{Sig}(\tau) \sim 2\varepsilon(1 + \mathrm{Re}\,\gamma(\tau))$

$V(\tau) = \text{visibility} = |\gamma(\tau)|$

$\tau_c = \int_{-\infty}^{\infty} |\gamma(\tau)|^2 d\tau$

$FT(\mathrm{Sig}(\tau)) \sim 2\varepsilon_0 \delta(\omega) + I(\omega) + I(-\omega)$

<u>Rays:</u> $\nabla R(\mathbf{r}) = n(\mathbf{r}) \hat{s}(\mathbf{r})$

<u>ABCD Matrices</u>

Translation: $\begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$

Flat surface refraction: $\begin{pmatrix} 1 & 0 \\ 0 & n_1/n_2 \end{pmatrix}$

Curved surface refraction:

$\begin{pmatrix} 1 & 0 \\ \frac{1}{R}(n_1/n_2 - 1) & n_1/n_2 \end{pmatrix}$

R = + for convex, – for concave

Spherical mirror/thin lens: $\begin{pmatrix} 1 & 0 \\ -1/f & 1 \end{pmatrix}$

$f_{lens} = ((n_2/n_1 - 1)(1/R_1 - 1/R_2))^{-1}$;

R = + for curving like "(", – for curving like ")"

$f_{mirror} = R/2$; $\ R$ = + for curving like ")"

Compound system, general properties:

$B = 0$ (image formation)

A = magnification

ABCD of d_O, then an optic, then d_i:

$\begin{pmatrix} A + d_i C & d_O A + B + d_O d_i C + d_i D \\ C & d_O C + D \end{pmatrix}$

$p_1 = (1 - D)/C, \ p_2 = (1 - A)/C, \ f = -1/C$

Cavity stability: $-1 < \frac{A + D}{2} < 1$

<u>Diffraction formulas</u>

Helmholtz Eq.: $\nabla^2 \mathbf{E}(\mathbf{r}) = -k^2 \mathbf{E}(\mathbf{r})$

Huygens-Fresnel: $E(x, y, z) = -\frac{i}{\lambda} \iint_{aper} E(x', y', 0) \frac{e^{ikR}}{R} dx' dy'$

Fresnel-Kirchhoff: $E(x, y, z) = -\frac{i}{\lambda} \iint_{aper} E(x', y', 0) \frac{e^{ikR}}{R} \left(\frac{1 + \cos(\mathbf{R}, \hat{\mathbf{z}})}{2} \right) dx' dy'$

Fresnel: $E(x, y, z) = -\frac{i e^{ikz} e^{ik(x^2 + y^2)/2z}}{\lambda z} \times \iint_{aper} E(x', y', 0) e^{ik(x'^2 + y'^2)/2z} e^{-ik(xx' + yy')/z} dx' dy'$

Fraunhofer: $E(x, y, z) = -\frac{i e^{ikz} e^{ik(x^2 + y^2)/2z}}{\lambda z} \times \iint_{aper} E(x', y', 0) e^{-ik(xx' + yy')/z} dx' dy'$

$I = I_0 |\text{2d FT of aperture function}|^2$

Single slit: $FT = \frac{1}{\sqrt{2\pi}}\, a\, \mathrm{sinc}(k_x a/2)$

Rectang.: $FT = \frac{1}{2\pi}\, ab\, \mathrm{sinc}(k_x a/2)\, \mathrm{sinc}(k_y b/2)$

Double: $FT = \sqrt{2/\pi}\ a\, \mathrm{sinc}(k_x a/2) \cos(k_x d/2)$

Top hat: $FT = \frac{a^2}{2} \frac{2 J_1(k_\rho a)}{k_\rho a} = \frac{a^2}{2} \mathrm{jinc}(k_\rho a)$

$k_x = kX/z, \ k_y = kY/z, \ k_\rho = k\rho/z$

<u>Spectrometer</u>: $\lambda = \frac{xh}{mz}, \ \Delta\lambda = \frac{\lambda}{mN}$

<u>Rayleigh</u>: $\theta_{min} \approx \frac{1.22\lambda}{D}$

Polarization: Jones and Stokes

LIGHT	Jones	Stokes
unpolar	n/a	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$
Horiz linear	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\0\\0\end{pmatrix}$
Vert linear	$\begin{pmatrix}0\\1\end{pmatrix}$	$\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}$
45° linear	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$
–45° linear	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$	$\begin{pmatrix}1\\0\\-1\\0\end{pmatrix}$
Linear at θ	$\begin{pmatrix}\cos\theta\\\sin\theta\end{pmatrix}$	$\begin{pmatrix}1\\\cos 2\theta\\\sin 2\theta\\0\end{pmatrix}$
RCP	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-i\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\1\end{pmatrix}$
LCP	$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}$	$\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$
Elliptical	$\begin{pmatrix}A\\Be^{i\delta}\end{pmatrix}$ with $A^2+B^2=1$	$\frac{1}{E_0^2}\begin{pmatrix}E_0^2\\E_{0x}^2-E_{0y}^2\\2E_{0x}E_{0y}\cos\delta\\2E_{0x}E_{0y}\sin\delta\end{pmatrix}$ δ is the phase shift of vert relative to horiz

Jones:

Angle of elliptical: $\alpha = \frac{1}{2}\tan^{-1}\left(\frac{2AB\cos\delta}{A^2-B^2}\right)$

$E_\alpha = |E_{eff}| \times \sqrt{A^2\cos^2\alpha + B^2\sin^2\alpha + AB\cos\delta\sin 2\alpha}$

$E_{\alpha\pm 90°} = |E_{eff}| \times \sqrt{A^2\sin^2\alpha + B^2\cos^2\alpha - AB\cos\delta\sin 2\alpha}$

Stokes:

Amount polarized: $\sqrt{S_1^2+S_2^2+S_3^2}$

Amount unpolarized: $S_0 - \sqrt{S_1^2+S_2^2+S_3^2}$

Deg of polarization $= \frac{\sqrt{S_1^2+S_2^2+S_3^2}}{S_0}$

OPTIC	Jones	Mueller (Stokes)
Horiz linear polar	$\begin{pmatrix}1&0\\0&0\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&1&0&0\\1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$
Vert linear polar	$\begin{pmatrix}0&0\\0&1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&-1&0&0\\-1&1&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$
45° linear polar	$\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&0&1&0\\0&0&0&0\\1&0&1&0\\0&0&0&0\end{pmatrix}$
–45° linear polar	$\frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&0&-1&0\\0&0&0&0\\-1&0&1&0\\0&0&0&0\end{pmatrix}$
angle θ linear polar	$\begin{pmatrix}\cos^2\theta&\sin\theta\cos\theta\\\sin\theta\cos\theta&\sin^2\theta\end{pmatrix}$	$\frac{1}{2}\begin{pmatrix}1&\cos 2\theta&\sin 2\theta&0\\\cos 2\theta&\cos^2 2\theta&\cos 2\theta\sin 2\theta&0\\\sin 2\theta&\cos 2\theta\sin 2\theta&\sin^2 2\theta&0\\0&0&0&0\end{pmatrix}$
$\lambda/4$ fast axis horiz	$\begin{pmatrix}1&0\\0&i\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix}$
$\lambda/4$ fast axis vert	$\begin{pmatrix}1&0\\0&-i\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&-1\\0&0&1&0\end{pmatrix}$
$\lambda/4$ fast axis 45°	$\frac{1}{2}\begin{pmatrix}1+i&1-i\\1-i&1+i\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&0&0&-1\\0&0&1&0\\0&1&0&0\end{pmatrix}$
$\lambda/4$ fast axis angle θ	$\begin{pmatrix}\cos^2\theta+i\sin^2\theta&(1-i)\sin\theta\cos\theta\\(1-i)\sin\theta\cos\theta&\sin^2\theta+i\cos^2\theta\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&\cos^2 2\theta&\frac{1}{2}\sin 4\theta&-\sin 2\theta\\0&\frac{1}{2}\sin 4\theta&\sin^2 2\theta&\cos 2\theta\\0&\sin 2\theta&-\cos 2\theta&0\end{pmatrix}$
$\lambda/2$ fast axis horiz	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$
$\lambda/2$ fast axis vert	$\begin{pmatrix}-1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&-1&0\\0&0&0&-1\end{pmatrix}$
$\lambda/2$ fast axis at θ	$\begin{pmatrix}\cos 2\theta&\sin 2\theta\\\sin 2\theta&-\cos 2\theta\end{pmatrix}$	$\begin{pmatrix}1&0&0&0\\0&\cos 4\theta&\sin 4\theta&0\\0&\sin 4\theta&-\cos 4\theta&0\\0&0&0&-1\end{pmatrix}$
Reflect	$\begin{pmatrix}-r_p&0\\0&r_s\end{pmatrix}$	(not worrying about)
Transmit	$\begin{pmatrix}t_p&0\\0&t_s\end{pmatrix}$	(not worrying about)
Rotation	$R=\begin{pmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{pmatrix}$ $M_{\substack{element\ with\\axis\ at\ angle}} = RM_{\substack{element\ with\\horizontal\ axis}}R^{-1}$	$R=\begin{pmatrix}1&0&0&0\\0&\cos 2\theta&-\sin 2\theta&0\\0&\sin 2\theta&\cos 2\theta&0\\0&0&0&1\end{pmatrix}$ $M_{\substack{element\ with\\axis\ at\ angle}} = RM_{\substack{element\ with\\horizontal\ axis}}R^{-1}$