Lecture 23: Fri, 29 Feb 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. "Quadratic dispersion" refers to:

a. k being a quadratic function of ω

b. ω being a quadratic function of k

Q2. The quantity in the dispersion that determines the broadening of a pulse is

a. the constant term

b. the linear term

c. the squared term

Q3. The book defines a quantity $T = \tau\sqrt{1+\Phi^2}$, where τ is the width of a Gaussian light pulse as it enters a bit of quadratic dispersion material and Φ is a complicated function relating to the phase of the pulse as it emerges from the material. What physical significance does T have?

a. It is a measure of the temporal width of the exiting light pulse.

b. It is a measure of the decrease in amplitude of the exiting light pulse.

c. Both ~~b~~ a and ~~c~~ b

Section 7.5 Group delay of wave packet

single plane wave $\longrightarrow$ picks up a phase shift $e^{i\vec{k}\cdot\Delta\vec{r}}$

Multiple Fourier components $\longrightarrow$ each component picks up a slightly different phase shift since k is different

$$\frac{\omega}{k} = \frac{c}{n} \rightarrow k = \frac{n}{c}\omega$$

(n is function of ω if there dispersion)

Plan: 1) Do FT of $E(t)$ to get freq spectrum $E(\omega, \vec{r}_0)$

2) Apply $e^{i\vec{k}\cdot\Delta\vec{r}}$ to each component to represent traveling by $\Delta\vec{r}$

3) Do inverse FT to figure out $E(t)$ at new position $\vec{r}_0 + \Delta\vec{r}$

Steps 2+3:
$$E(t, \vec{r}_0 + \Delta\vec{r}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[E(\omega, \vec{r}_0) e^{i\vec{k}\cdot\Delta\vec{r}} \right] e^{-i\omega t} d\omega$$

Taylor series: $\vec{k}\cdot\Delta\vec{r} \simeq \vec{k}\Big|_{\omega_0}\cdot\Delta\vec{r} + \frac{\partial\vec{k}}{\partial\omega}\Big|_{\omega_0}(\omega-\omega_0)\cdot\Delta\vec{r} + \dots$

around $\omega = \omega_0$ "carrier freq" $k = f(\omega)$

$$E(t, \vec{r}_0 + \Delta\vec{r}) = \frac{1}{\sqrt{2\pi}} e^{i\left(\vec{k}(\omega_0)\cdot\Delta\vec{r} - \omega_0 \frac{\partial\vec{k}}{\partial\omega}(\omega_0)\cdot\Delta\vec{r}\right)} \int_{-\infty}^{\infty} E(\omega, \vec{r}_0) e^{-i\omega\left(t - \frac{\partial\vec{k}}{\partial\omega}(\omega_0)\cdot\Delta\vec{r}\right)} d\omega$$

Define $t' = \frac{\partial\vec{k}}{\partial\omega}(\omega_0)\cdot\Delta\vec{r}$

$$= \frac{1}{\sqrt{2\pi}} e^{i\left(\vec{k}(\omega_0)\cdot\Delta\vec{r} - \omega_0 t'\right)} \int_{-\infty}^{\infty} E(\omega, \vec{r}_0) e^{-i\omega(t-t')} d\omega$$

Approx: if $\vec{k}$ changes only in real part

then $t' = \frac{\partial}{\partial\omega} Re(\vec{k})\Big|_{\omega=\omega_0}\cdot\Delta\vec{r}$ = real

and integral is basically Inverse FT, with time $= t - t'$

$$= e^{i\left(\vec{k}(\omega_0)\cdot\Delta\vec{r} - \omega_0 t'\right)} E(t - t', \vec{r}_0)$$

overall phase shift.

Same as wave travelling at

$v = \frac{\omega_0}{k(\omega_0)}$ = phase velocity of ω_0 phase

if $\vec{k}$ = complex at ω_0

$$E(t, \vec{r}_0 + \Delta\vec{r}) = e^{i(k_{real}(\omega_0)\cdot\Delta r + i\,k_{imag}(\omega_0)\cdot\Delta r - \omega_0 t)} \; E(t-t', \vec{r}_0)$$

$$= e^{-\{k_{imag}(\omega_0)\cdot\Delta r} \; e^{i(stuff)} \; E$$

$$I \propto |E|^2 = e^{-2k_{imag}(\omega_0)\cdot\Delta r} \; |E(t-t', \vec{r}_0)|^2$$

$\downarrow$

absorption may have reduced pulse intensity

t' = "group delay function"

$\boxed{\text{End result: pulse delayed + reduced}}$

Group velocity $\frac{1}{v_g} = \frac{\partial k}{\partial \omega}$

$$= \frac{\partial\, Re(k)}{\partial \omega}\Big|_{\omega=\omega_0}$$

since the imaginary part of k is not changing (much)

Then $t' = \frac{1}{v_g}\Delta r$ (w/o dot product)

Note: higher order terms in Taylor series $\vec{k}\cdot\Delta\vec{r}$ can become important (next section)

$\rightarrow$ $\boxed{\text{pulse spreads out}}$

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Section 7.6 Quadratic Dispersion

specifically $E(t) = \text{[illegible]} = E_0 e^{-t^2/\tau^2} e^{-i\omega_0 t}$

the $E(\omega) = E_0 e^{-\tau^2(\omega-\omega_0)^2/2}$

$k = K\hat{z}$

and $\vec{\Delta r} = z\hat{z} \rightarrow \vec{K}\cdot\vec{\Delta r} = kz$

$\vec{k} \rightarrow$ | | $+z$

a prime quadratic k_0 k_1 k_2 k_3 k_4

we'll [illegible] this lecture/lesson

Next term in expansion

Then $\vec{K}\cdot\vec{\Delta r}$ expansion is ...

$$kz = K\Big|_{\omega=\omega_0} z + \underbrace{\frac{\partial k}{\partial \omega}\Big|_{\omega=\omega_0}}_{\text{defined as } \frac{1}{v_g}} (\omega-\omega_0)\, z + \underbrace{\frac{\partial^2 k}{\partial \omega^2}\Big|_{\omega=\omega_0}}_{\alpha \text{ (including } 2!)} \frac{1}{2!}(\omega-\omega_0)^2 z$$

$$\boxed{kz = k_0 z + \frac{1}{v_g}(\omega-\omega_0)z + \alpha(\omega-\omega_0)^2 z}$$

where k_0 = k evaluated at ω_0

in terms of n, if $n(\omega)$ is known

use $k = \frac{1}{c} n(\omega)\omega$

$$\frac{\partial k}{\partial \omega} = \frac{1}{c}(n'\omega + n) \quad \text{product rule}$$

$$\frac{\partial^2 k}{\partial \omega^2} = \frac{1}{c}(n''\omega + n' + n')$$

$$\boxed{kz = \underbrace{\left(\frac{1}{c}\omega_0 n(\omega_0)\right)}_{k_0} z + \underbrace{\frac{1}{c}\left(\omega_0 n'(\omega_0) + n(\omega_0)\right)}_{\frac{1}{v_g}}(\omega-\omega_0)z + \underbrace{\frac{1}{2c}\left(\omega_0 n''(\omega_0) + 2n'(\omega_0)\right)}_{\alpha}(\omega-\omega_0)^2 z}$$

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Back to 3-step pln, for $\boxed{\text{Gaussian wavepacket}}$ $E(t) = E_0 e^{-t^2/2\tau^2} e^{-i\omega_0 t}$

(1) FT of $E(t)$: $E(\omega) = \tau E_0 e^{-\tau^2(\omega-\omega_0)^2/2}$

(2) + (3): $E(t, z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[E(\omega, \vec{r}_0) e^{i\vec{k}\cdot\Delta\vec{r}} \right] e^{-i\omega t} d\omega$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(\tau E_0 e^{-\tau^2(\omega-\omega_0)^2/2} \right) e^{i\left(k_0 z + \frac{1}{v_g}(\omega-\omega_0) z + \alpha(\omega-\omega_0)^2 z\right)} e^{-i\omega t} d\omega$$

let $\omega' = \omega - \omega_0$

$\omega = \omega' + \omega_0$

$$= \frac{1}{\sqrt{2\pi}} \tau E_0 e^{i(k_0 z - \omega_0 t)} \int_{-\infty}^{\infty} e^{-\tau^2 \omega'^2/2} e^{i\left(\frac{\omega'}{v_g} z + \alpha z \omega'^2 - \omega' t\right)} d\omega'$$

$$\underbrace{\int_{-\infty}^{\infty} e^{-\left(\frac{\tau^2}{2} + i\alpha z\right)\omega'^2 + i\left(\frac{z}{v_g} - t\right)\omega'} d\omega'}_{= \sqrt{\frac{\pi}{\left(\frac{\tau^2}{2} - i\alpha z\right)}} e^{i^2 \left(\frac{z}{v_g} - t\right)^2 / 4\left(\frac{\tau^2}{2} - i\alpha z\right)}}$$

using Eq. 10.90 integral table

$$= \frac{1}{\sqrt{2}} \tau E_0 e^{i(k_0 z - \omega_0 t)} \underbrace{\sqrt{\frac{1}{\frac{\tau^2}{2}\left(1 - \frac{2i\alpha z}{\tau^2}\right)}} \times \frac{1 + \frac{2i\alpha z}{\tau^2}}{1 + \frac{2i\alpha z}{\tau^2}}} \underbrace{e^{-\frac{\left(t - \frac{z}{v_g}\right)^2}{2\tau^2\left(1 - \frac{2i\alpha z}{\tau^2}\right)}} \times \frac{1 + \frac{2i\alpha z}{\tau^2}}{1 + \frac{2i\alpha z}{\tau^2}}}$$

cancel

$$\sqrt{\frac{2}{\tau^2}} \frac{1}{\sqrt{1^2 + \left(\frac{2\alpha z}{\tau^2}\right)^2}} \sqrt{1 + \frac{2i\alpha z}{\tau^2}} \quad \exp\left[\frac{-\left(t - \frac{z}{v_g}\right)^2}{2\tau^2} \frac{\left(1 + \frac{2i\alpha z}{\tau^2}\right)}{1^2 + \left(\frac{2\alpha z}{\tau^2}\right)^2} \right]$$

what is sqrt $(1 + ix)$?

$$1 + ix = \sqrt{1 + x^2}\, e^{i\tan^{-1} x}$$

$$\text{so } \sqrt{} = (1 + x^2)^{1/4} e^{\frac{i}{2}\tan^{-1}(x)}$$

Let $\boxed{\Phi = \frac{2\alpha z}{\tau^2}}$

$$= E_0 e^{i(k_0 z - \omega_0 t)} \frac{1}{\sqrt{1 + \Phi^2}} (1 + \Phi^2)^{1/4} e^{\frac{i}{2}\tan^{-1}\Phi} \exp\left[-\frac{\left(t - \frac{z}{v_g}\right)^2}{2\tau^2} \frac{(1 + i\Phi)}{1 + \Phi^2} \right]$$

Let $\boxed{T^2 = \tau^2(1 + \Phi^2)}$ $\quad T = \tau\sqrt{1 + \Phi^2}$

$$\boxed{E(t, z) = \frac{E_0 e^{i(k_0 z - \omega_0 t)}}{(1 + \Phi^2)^{1/4}} \exp\left[\frac{i}{2}\tan^{-1}\Phi - \frac{i}{2} \frac{\Phi}{T^2}\left(t - \frac{z}{v_g}\right)^2 \right] e^{-\frac{1}{2}\frac{1}{T^2}\left(t - \frac{z}{v_g}\right)^2}}$$

↳ could be expressed as $\sqrt{T/\tau}$

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Notes • Φ and T are functions of z

• when $z=0$, we get $\Phi=0$, $T=\tau$

$$E(t,z) = \frac{E_0 e^{-i\omega_0 t}}{1} \exp\left[0 - 0 - \frac{1}{2}\frac{1}{\tau^2}(t)^2\right]$$

$$= E_0 e^{-t^2/2\tau^2} e^{-i\omega_0 t}$$

= input pulse ✓

• pulse moves at speed v_g !

proof: pulse peak goes when $t = \frac{z}{v_g}$ → $z = v_g t$

(other terms in exponential are phase factors)

• width of pulse ~ T

T increases w/ Φ and Φ increases with z

→ pulse width increases with z !

• overall amplitude ~ $\frac{1}{(1+\Phi^2)^{1/4}}$

amplitude can also be reduced if have some imaginary component to k_0.

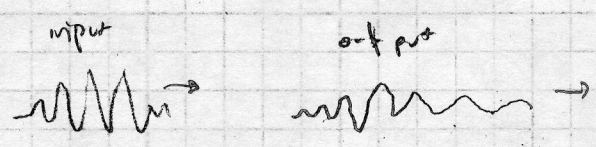
Φ increases with z

→ overall amplitude decreases with z !

• pulse "chirps" (this problem, unassigned)

low frequencies (eg red) get ahead of high freqs (eg blue)

since red has a lower n



oscillations plotted in space

(in time: backwards)

"chirp" from sound waves

These results more general than the Gaussian wavepacket case studied here, but even more cumbersome to analyze for other waveshapes

Skip Section 7.7

Lecture 24: Mon, 3 Mar 2008

Reading quizzes: no talking, no looking in your books/notes

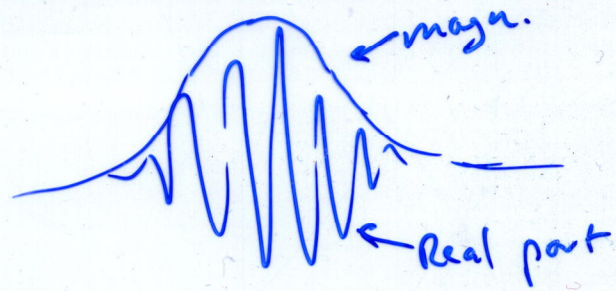
Q1. The interferometer studied was the

a. Fresnel
b. Michelson
c. Morley
d. Young

Q2. T/F: In a typical interferometer setup, the electric field at the detector is the sum of *two* electric fields, one from each path.

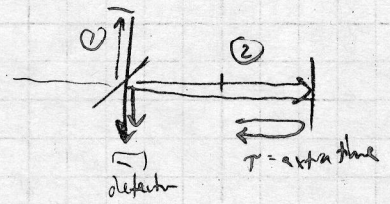
Q3. T/F: In a typical interferometer setup, the "degree of coherence function", $\gamma(\tau)$, typically *oscillates* as τ is shifted from zero.

(Real part)



Chapter 8

Michelson interferometer



$$I_{detector} = \frac{1}{2}\frac{1}{\mu_0 c}\left|\left(E_{path\,1} + E_{path\,2}\right)\right|^2$$

$$= \frac{1}{2}\frac{1}{\mu_0 c}\left|E_0 e^{i(kz-\omega t)} + E_0 e^{i(kz-\omega(t-\tau))}\right|^2$$

at $t=0$ it's a max

at time $t=\tau$ it's a max

if single ω planewaves (equal magnitude)

$$= \underbrace{\frac{1}{2}\frac{1}{\mu_0 c}\left|E_0\right|^2}_{I_0}\ \underbrace{\left|e^{i(kz-\omega t)}\right|^2}_{1}\ \left|1 + e^{i\omega\tau}\right|^2$$

$$(1+e^{i\omega\tau})(1+e^{-i\omega\tau})$$

$$1 + \underbrace{e^{i\omega\tau} + e^{-i\omega\tau}}_{2\cos\omega\tau} + 1$$

$$\boxed{I_{det} = I_0\, 2\left(1+\cos\omega\tau\right)}$$

I_0 = path 1 only

What happens when ω = frequency band instead of single frequency?
e.g. from pulse instead of infinite plane wave.

More general eqn.:

$$I_{det} = \frac{1}{2}\frac{1}{\mu_0 c}\underbrace{\left|E(t) + E(t-\tau)\right|^2}_{\left(E(t)+E(t-\tau)\right)\cdot\left(E^*(t)+E^*(t-\tau)\right)}$$

$$|E(t)|^2 + |E(t-\tau)|^2 + E(t)\,E^*(t-\tau) + c.c.$$

$$x + x^* = 2\,\mathrm{Re}(x)$$

$$I_{det} \simeq I(t) + I(t-\tau) + \frac{1}{\mu_0 c}\,\mathrm{Re}\left(E(t)\,E^*(t-\tau)\right)$$

total energy $\sim \int I\,dt$

We want to integrate all terms. But first, focus on last term. Integral is an "auto correlation" kind of like convolution but same function
- $t-\tau$ instead of $\tau - t$
- complex conjugate

$$\text{Thm: } FT\left[\underbrace{\int_{-\infty}^{\infty} E(t)\,E^*(t-\tau)\,dt}_{\text{function of }\tau}\right] = \sqrt{2\pi}\,\left|FT(E(t))\right|^2$$

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$$\int_{-\infty}^{\infty} I_{det}\, dt = \underbrace{\int_{-\infty}^{\infty} I(t)\, dt + \int_{-\infty}^{\infty} I(t-\tau)\, dt}_{\text{same}} + \frac{1}{\mu_0 c} \mathrm{Re}\left[\mathrm{Imag\ FT} \quad \underbrace{\mathrm{FT}\left[\int_{-\infty}^{\infty} E(t) E^*(t-\tau)\, dt\right]}_{\sqrt{2\pi}\,|\mathrm{FT}(E(t))|^2 \text{ by thm}} \right]$$

wrt τ

(do $t' = t-\tau$)

$\sqrt{2\pi}\,|E(\omega)|^2$

Parseval's thm: $\int_{\mathbb{R}} I(t)\,dt = \int_{\mathbb{R}} I(\omega)\,d\omega$

recall $I(\omega) = \frac{1}{2}\frac{1}{\mu_0 c}|E(\omega)|^2$

$$\mathrm{Re}\left[\frac{1}{\cancel{\sqrt{2\pi}}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau} \left(\cancel{\sqrt{2\pi}} \cdot 2\mu_0 c\, I(\omega)\right)\right]$$

cancels

$$= 2\int_{-\infty}^{\infty} I(\omega)\, d\omega + 2\,\mathrm{Re}\left[\int_{-\infty}^{\infty} I(\omega)\, e^{-i\omega\tau}\, d\omega\right]$$

define $\boxed{\gamma = \dfrac{\int_{-\infty}^{\infty} I(\omega) e^{-i\omega\tau}\, d\omega}{\int_{-\infty}^{\infty} I(\omega)\, d\omega}}$ = Degree of coherence (function of τ)

$$\boxed{\int_{-\infty}^{\infty} I_{det}(t,\tau)\, dt = 2\underbrace{\left[\int_{-\infty}^{\infty} I(\omega)\, d\omega\right]}_{\text{"}\mathcal{E}_0\text{"} \equiv \text{total energy (per area) of pulse}}\left[1 + \mathrm{Re}(\gamma(\tau))\right]}$$

what about cw light? integrals diverge (maybe)

then use average intensity $\langle I \rangle = \frac{1}{T}\int_0^T I(t)\, dt$

T large enough so $\langle I \rangle$ = constant

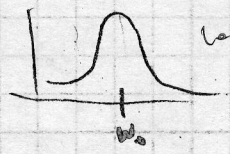
i.e. cut of integral instead of going to infinity

focus on a time T, look at integrand

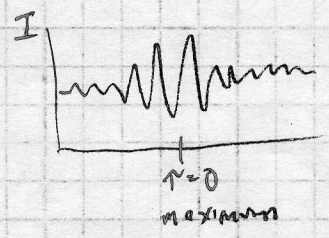
$$\langle I_{det}(\tau) \rangle = 2\langle I \rangle \left[1 + \mathrm{Re}(\gamma(\tau))\right]$$

γ = same thing as above,

$I(\omega)$ = experimental measurements

HW problem: $I(\omega) =$  Lorentzian shape

What does intensity do as you send this (cw) light through interferometer and vary τ?

Answer: 

note $\mathrm{Re}(\gamma)$ looks very similar since $I \sim 1 + \mathrm{Re}(\gamma)$

$|\gamma|$ = envelope of oscillations

Some terms

Coherence time: the amount of delay (τ) needed to cause I to stop oscillating

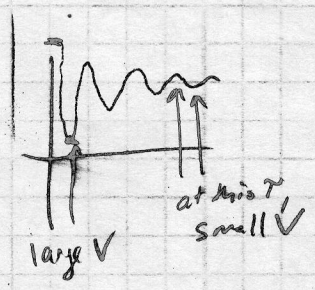
one measure: $\tau_c = \int_{-\infty}^{\infty} |\gamma(\tau)|^2 \, d\tau$

i.e. the more area, the longer you'll have to wait to have no oscillations

Coherence length: $l_c = c\,\tau_c$ distance associated w/ τ_c. clear enough.

Fringe visibility: $V = \dfrac{I_{max} - I_{min}}{I_{max} + I_{min}}$ of nearby oscillations

(or integrated intensity, if pulses)



Monday - Stephen

HW problem: $V = |\gamma(\tau)|$

Mid-beginning cw light

For $\boxed{\text{Gaussian wave packet}}$

$E(t) = E_0 e^{-t^2/2\sigma^2} e^{-i\omega_0 t}$

$E(\omega) = E_0 \tau e^{-\tau^2(\omega-\omega_0)^2/2}$

can't use τ! already using for delay $\tau \Rightarrow \frac{1}{\Delta\omega}$

$I(\omega) = \frac{n}{2\mu_0 c} |E(\omega)|^2 = I_0 e^{-\tau^2(\omega-\omega_0)^2}$

$= I_0 e^{-\frac{(\omega-\omega_0)^2}{\Delta\omega^2}}$

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