

Lecture 33: Mon, 24 Mar 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. The _____ approximation is a special case of the _____ approximation

- a. Fraunhofer; Fresnel
- b. Fresnel; Fraunhofer

Q2. T/F: The Fraunhofer approximation predicts a diffraction pattern that changes in shape as distance from aperture is increased.

Q3. To calculate the Fraunhofer diffraction pattern for a given aperture, you must calculate:

- a. a 2-D Fourier transform
- b. a correlation function
- c. a Green's surface

The or maybe "a"
 Soln to this eqn: (His problem)

$$\tilde{E} = -\frac{i}{\lambda z} \iint_{\text{aperture}} \tilde{E}(x', y') e^{i \frac{k}{2z} [(x-x')^2 + (y-y')^2]} dx' dy'$$

do integral, then $E = \tilde{E} e^{ikz}$

field strength across aperture (if varying) ← "aperture function"

stop here in Fri
 stop here in Mon

gives answer just like our original guess with

$$r = [(x-x')^2 + (y-y')^2 + z^2]^{1/2}$$

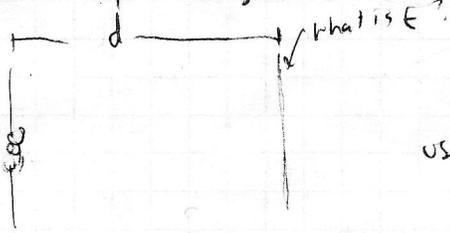
$$= z \left[1 + \frac{(x-x')^2}{z^2} + \frac{(y-y')^2}{z^2} \right]^{1/2}$$

$$= z \left(1 + \frac{1}{2} \frac{(x-x')^2}{z^2} + \frac{1}{2} \frac{(y-y')^2}{z^2} + \dots \right)$$

but with proper $-\frac{i}{\lambda}$ out in front

just a keep zeroth order for denom. ($\frac{1}{z}$)
 keep first order for exponent (need that since small change in x → big change in e^{ikx})

To actually do integral



use $(x-x')^2 = x^2 - 2xx' + x'^2$
 can be taken out of integral

(same for y)

$$E = -\frac{i}{\lambda d} e^{ikz} e^{i \frac{k}{2d}(x^2+y^2)} \iint_{\text{aperture}} \tilde{E}(x', y') e^{i \frac{k}{2d}(x'^2+y'^2)} e^{-i \frac{k}{d}(xx'+yy')} dx' dy'$$

"The Fresnel Approximation"

Babinet: if aperture =

(square lets light through circle blocks it)

then $\iint_{\square} () da' = \iint_{\circ} () da'$

Not so interesting?

diffraction from hair $\iint_{\text{hair}} = \iint_{\text{all space}} - \iint_{\text{slit}}$ → hair diffraction very similar to slit!

from → diffraction like = (we'll see later)

bright spot in middle! "Poisson's spot"

Section 10.5 Fraunhofer Approx

$$e^{i \frac{k}{z} (x'^2 + y'^2)} \approx 1$$

True if $d \rightarrow \infty$ "far field"

What about $e^{-i \frac{k}{d} (xx' + yy')}$ term? Not also = 1?

→ Not necessarily: x' and y' limited by size of aperture
 x and y have no such limit.

Result

$$E \approx -\frac{i}{\lambda d} e^{i k d} e^{i \frac{k}{z} (x^2 + y^2)} \iint_{\text{aperture}} \tilde{E}(x', y') e^{-i \frac{k}{d} (xx' + yy')} dx' dy'$$

define $k_x = \frac{kx}{d}$ $k_y = \frac{ky}{d}$ $k = \frac{2\pi}{\lambda}$
 normal wavevector

$$\iint \tilde{E}(x', y') e^{-i k_x x'} e^{-i k_y y'} dx' dy'$$

2D Fourier Transform of aperture function! (w/o 2π)

(or inverse FT to be technical)

x' like t
 k_x like ω

We mainly want intensity, $I \propto |E|^2$

Note: exponential factors come from $\cos(kx - \omega t) = e^{i(kx - \omega t)}$
 but $\cos(\omega t - kx) = e^{i(\omega t - kx)}$
 so everything must be unchanged (I think!) if $t \rightarrow -t$
 $x \rightarrow -x$
 $y \rightarrow -y$

$$I \propto |2D \text{ F.T. of aperture function}|^2$$

Then it really is FT.

important result!

often aperture function $\begin{cases} 1 & \text{inside aperture} \\ 0 & \text{outside aperture} \end{cases}$
 can be otherwise, e.g. filter over part of aperture

2D Fourier Transform

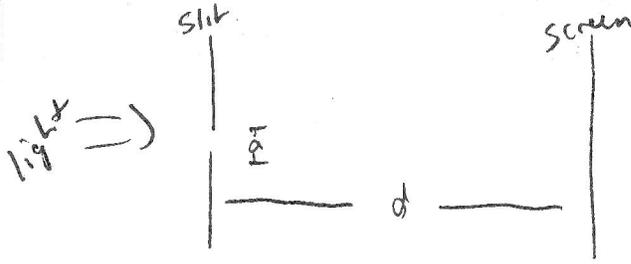
often our aperture function will be separable

$$E(x', y') = E(x') \cdot E(y')$$

then

$$\begin{aligned} & \iint E(x', y') e^{ik_x x'} e^{ik_y y'} dx' dy' \\ &= \int E(x') e^{ik_x x'} dx' \int E(y') e^{ik_y y'} dy' \\ &= \text{FT}(x \text{ function}) \times \text{FT}(y \text{ function}) \end{aligned}$$

Application: Single Slit



Aperture function
AF

$$FT(\text{rect}) = \int_{-a/2}^{a/2} e^{ikx} dx = \frac{1}{ik} e^{ikx} \Big|_{-a/2}^{a/2} = \frac{1}{ik} (e^{ika/2} - e^{-ika/2}) = \frac{2 \sin(ka/2)}{k}$$

What is diffraction pattern of a single slit?

If it's long + thin, we can neglect the Y-dimension

$$E \propto \hat{y}(AF)$$

~~this AF is the "long + thin" case~~
~~instead of y, scaled to width a, so~~

$$E = E_0 \text{sinc}\left(\frac{k_x a}{2}\right)$$

X, Y = coordinates on screen

Using defn of $k_x = \frac{kX}{d}$

$$I(X) = I_0 \text{sinc}^2\left(\frac{k a X}{d}\right) \rightarrow \text{this is the pattern you see!}$$

If it's rectangular, but not long enough to neglect the Y-dimension, you just get another sinc for Y:

$$I(x) = I_0 \text{sinc}^2\left(\frac{k a X}{d}\right) \text{sinc}^2\left(\frac{k b Y}{d}\right)$$

(b is length of slit in Y-dimension)

Plots of this are on the next 2 pages.

long + thin $\rightarrow \delta$ function in Y direction

Diffraction from a rectangular slit

John S. Colton

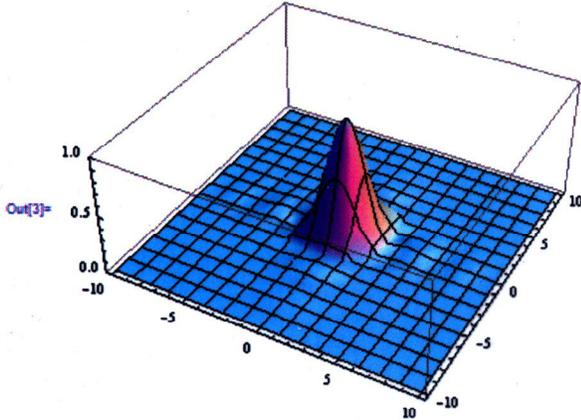
In[1]:= `sinc[x_] = Sin[x] / x`

Out[1]=
$$\frac{\text{Sin}[x]}{x}$$

In[2]:= `intensity[x_, y_, a_, b_] = sinc[a x]^2 sinc[b y]^2`

Out[2]=
$$\frac{\text{Sin}[a x]^2 \text{Sin}[b y]^2}{a^2 b^2 x^2 y^2}$$

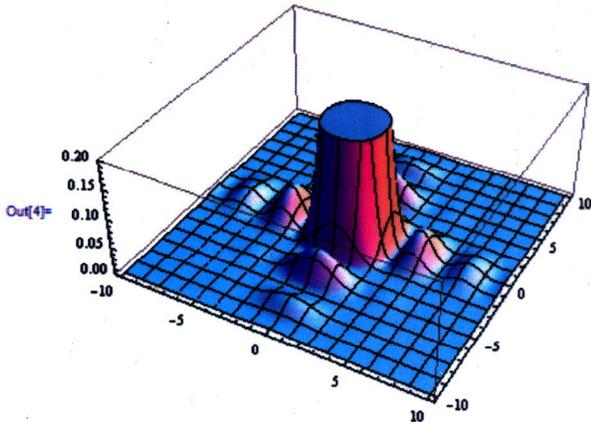
In[3]:= `Plot3D[intensity[x, y, 1, 1], {x, -10, 10}, {y, -10, 10}, PlotRange -> {0, 1}]`



Square slit: $a = b$



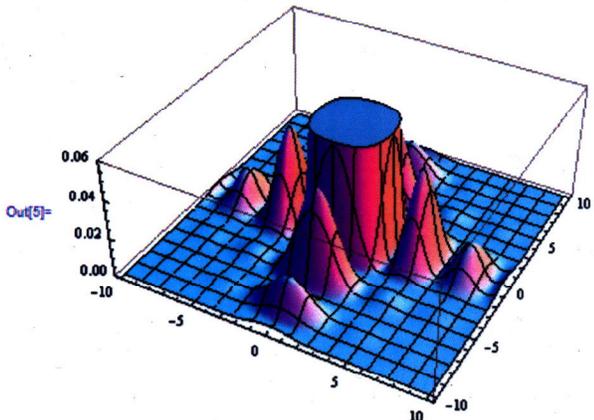
In[4]:= `Plot3D[intensity[x, y, 1, 1], {x, -10, 10}, {y, -10, 10}, PlotRange -> {0, .2}, PlotPoints -> 100]`



zoomed in a bit

$a = b$

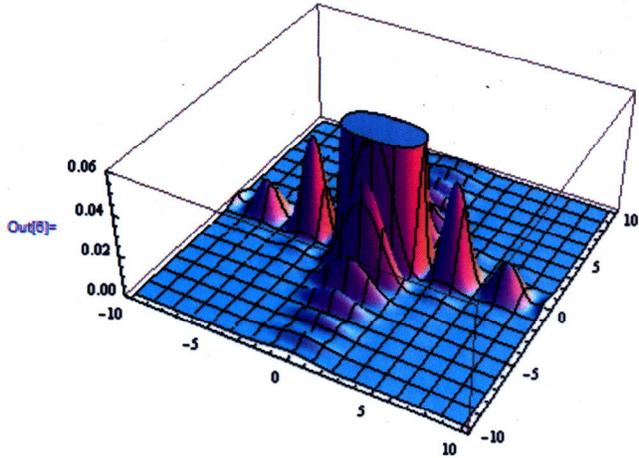
In[5]:= `Plot3D[intensity[x, y, 1, 1], {x, -10, 10}, {y, -10, 10}, PlotRange -> {0, .06}, PlotPoints -> 100]`



zoomed in further
side peaks are now quite evident

$a = b$

```
In[6]:= Plot3D[intensity[x, y, 1, 2], {x, -10, 10}, {y, -10, 10},
  PlotRange -> {0, .06}, PlotPoints -> 100]
```

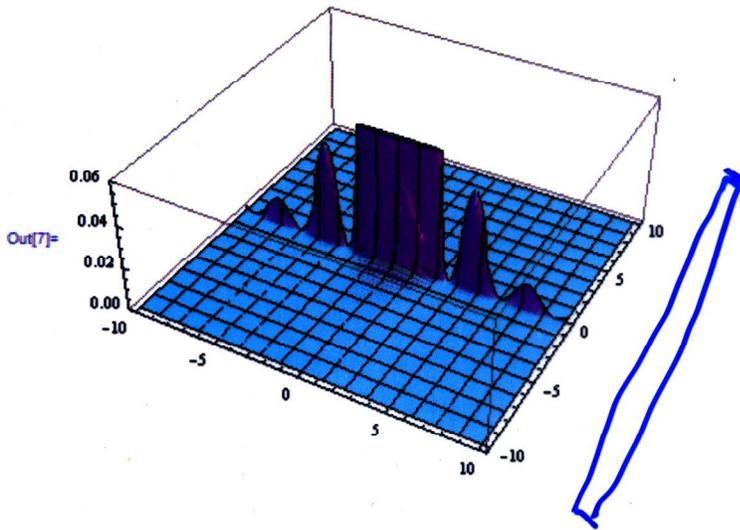


Rectangular: $b = 2a$

Still zoomed in a lot

Side lobes in y-direction spaced closer together than in x-direction

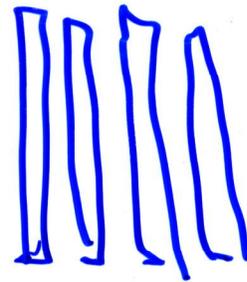
```
In[7]:= Plot3D[intensity[x, y, 1, 25], {x, -10, 10}, {y, -10, 10},
  PlotRange -> {0, .06}, PlotPoints -> 250]
```



Rectangular: $b = 25a$

Very, very sharp in the y-direction, like a delta function. (The side lobes now spaced so closely together you may not be able to see them.) All of the visible diffraction pattern is in the x-dimension. This is what you would normally consider "diffraction by a slit" to be.

of $I \rightarrow \dots \rightarrow \frac{K^2}{2} \frac{1}{\sin^2 \theta} \dots$



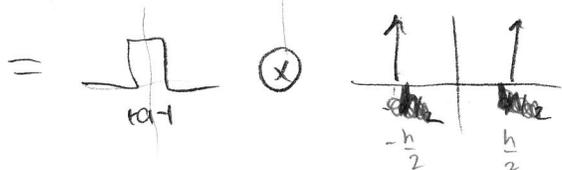
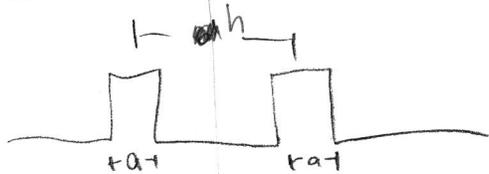
Application of the Convolution Theorem

The convolution theorem can make some problems trivial. For example:

Double Slit



1D Aperture Function (ie other dimension = wide)



$$\mathcal{F}(\omega) = \mathcal{F}(a) \times \mathcal{F}(h)$$

$$= \frac{1}{\sqrt{2\pi}} a \operatorname{sinc} \frac{k_x a}{2} \times \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (1) e^{ik_x x} dx$$

as done previously

$$k_x = \frac{kx}{d}$$

$$e^{-ik_x h/2} + e^{+ik_x h/2} = 2 \cos k_x \frac{h}{2}$$

$$\mathcal{F}(k_x) = \frac{1}{2\pi} 2a \operatorname{sinc} \frac{k_x a}{2} \cos \frac{k_x h}{2}$$

zeros/maxima of this probably given in 123

That should look familiar to those who remember the lab!

Piece of cake!

$$I = \left(\frac{1}{d}\right)^2 I_0 \operatorname{sinc}^2 \frac{k_x a}{2} \cos^2 \frac{k_x h}{2}$$

↑
technically

double slit pattern!



(h x a so cos oscillates faster)