Lecture 36: Mon, 31 Mar 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. The Rayleigh criterion for image resolution is (using the "jinc" function):

a. The second peak's maximum should be located at least as far away as the first peak's *first minimum*.
b. The two peaks should be separated by at least the first peak's *full width* at half maximum
c. The two peaks should be separated by at least the first peak's *half width* at half maximum

Q2. A lens placed after a diffraction aperture causes the Fraunhofer pattern to be produced:

a. before the focal length
b. at the focal length
c. after the focal length (but at finite distance)
d. at infinity

Q3. A Gaussian-shaped beam, when diffracting through an "infinitely large aperture", turns into:

a. another Gaussian-shaped beam
b. a $(\text{jinc})^2$-shaped beam
c. a Lorentzian-shaped beam
d. a $(\text{sinc})^2$-shaped beam

problem cont. from last time –

$$F.T. = \underbrace{e^{ikx\,2a}}_{\text{shift of square}} \underbrace{\left[a\,\text{sinc}\left(\frac{k_x a}{2}\right)\right]\left[a\,\text{sinc}\left(\frac{k_y a}{2}\right)\right]}_{\text{FT of square}}$$

$$+ \underbrace{e^{ikx(-2a)}}_{\text{shift of circle}} \underbrace{\left[2\pi a^2 \frac{J_1(k_r a)}{k_r a}\right]}_{\text{FT of circle}}$$

$$F.T. = e^{-ikx(2a)}\, a^2 \left[e^{i4k_x a}\,\text{sinc}\frac{k_x a}{2}\,\text{sinc}\frac{k_y a}{2} + 2\pi \frac{J_1(k_r a)}{k_r a}\right]$$

$$\text{So } I = I_0 \left| e^{i\frac{4kXa}{d}}\,\text{sinc}\frac{kXa}{2d}\,\text{sinc}\frac{kYa}{2d} + 2\pi \frac{J_1\left(k\sqrt{x^2+y^2}\,\frac{a}{d}\right)}{k\sqrt{x^2+y^2}\,\frac{a}{d}} \right|^2$$

complicated, but doable!

See plot on next page

Intensity of square-circle combo

John S. Colton

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In[1]:= r[x_, y_] = Sqrt[x^2 + y^2]
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Out[1]= $\sqrt{x^2 + y^2}$

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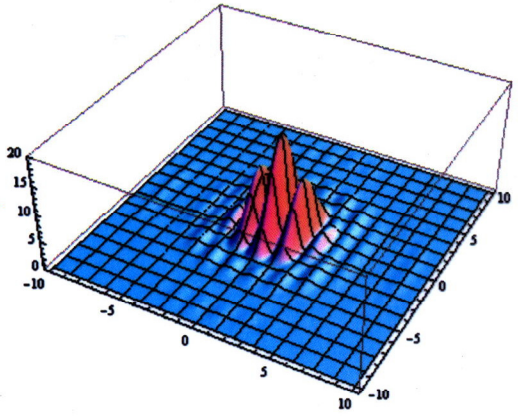
intensity[x_, y_] =
 (Abs[E^(I 4 x) Sinc[x / 2] Sinc[y / 2] + 2 Pi BesselJ[1, r[x, y]] / r[x, y]])^2
```

I made k = a = d = 1

Out[2]= $\text{Abs}\left[\frac{2\pi\,\text{BesselJ}\left[1, \sqrt{x^2+y^2}\right]}{\sqrt{x^2+y^2}} + e^{4ix}\,\text{Sinc}\left[\frac{x}{2}\right]\text{Sinc}\left[\frac{y}{2}\right]\right]^2$

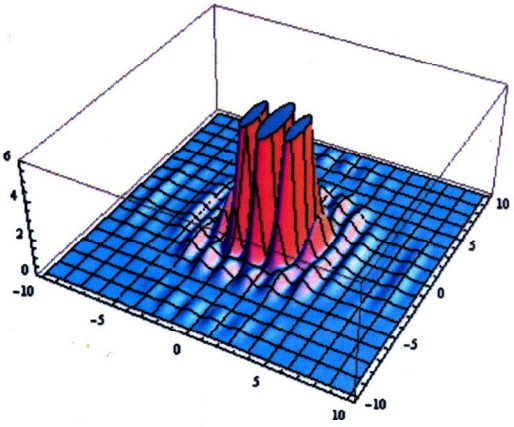
```
In[3]:= Plot3D[intensity[x, y], {x, -10, 10}, {y, -10, 10}, PlotRange → {0, 20},
 PlotPoints → 100]
```

Out[3]=



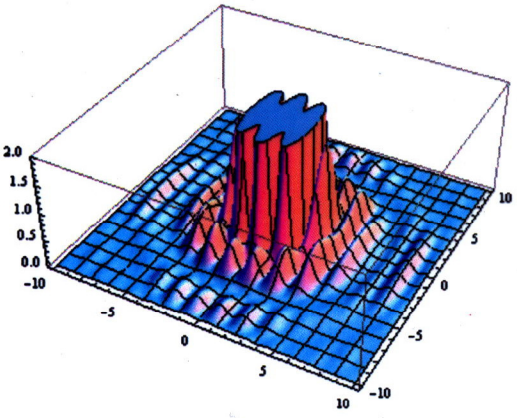
You see some interesting symmetries!
Some square, some circular

Out[7]=

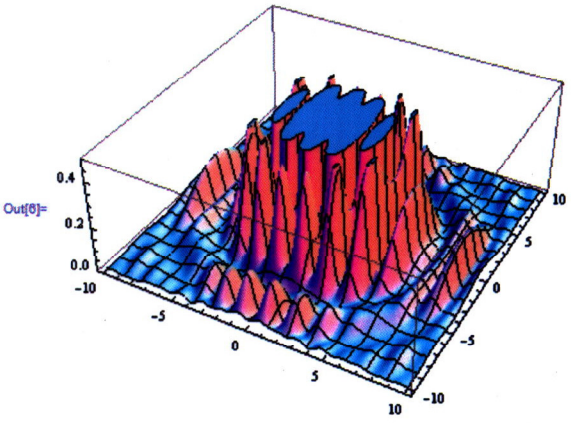


Increasing zoom

Out[5]=

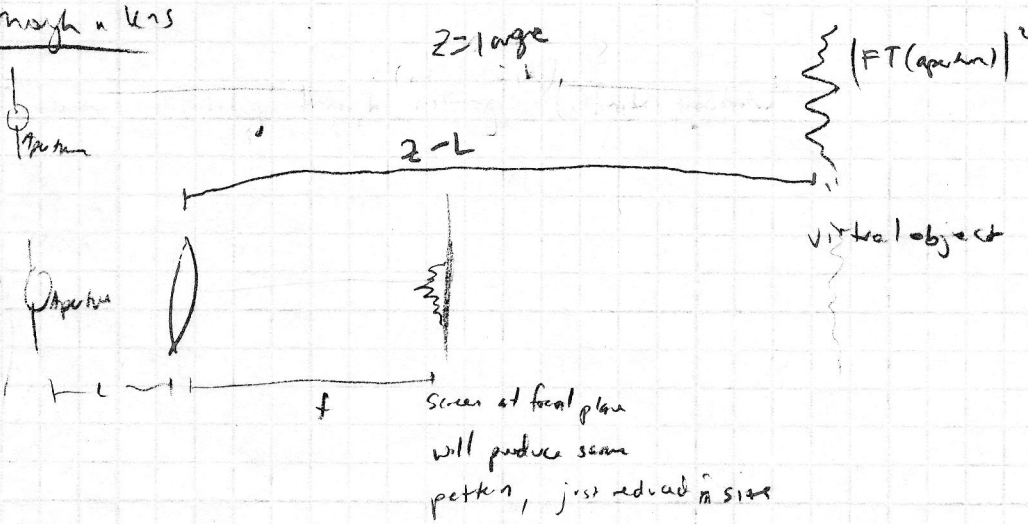


Out[6]=



Section 11.4

Diffraction through a lens



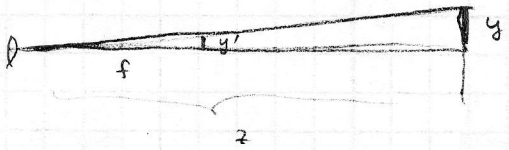
$$\frac{1}{f} = \frac{1}{o} + \frac{1}{i} \qquad o = -(z-L) \quad \text{virtual object at previous } z = \text{large}$$

$$\frac{1}{f} = \frac{1}{-(z-L)} + \frac{1}{i}$$

$\frac{1}{-(z-L)} \simeq 0$ since $z = \infty$

$i \simeq f$

$$M = -\frac{i}{o} = \frac{-f}{-(z-L)} \simeq \frac{f}{z}$$



$$\frac{y'}{y} = \frac{f}{z} \rightarrow \frac{y'}{f} = \frac{y}{z} \simeq \theta$$

Angular size preserved!

Skip lots of work in book ... end result for screen at lens's focal length

Old Fraunhofer: $E(x,y,z=d) \overset{\text{large}}{\simeq} -\frac{i}{\lambda d} e^{ikd} e^{\frac{ik r^2}{2d}} FT(\text{aperture})$ with $r = \sqrt{x^2+y^2}$

$k_x = \frac{kx}{d}$ $k_y = \frac{ky}{d}$

New result: $E(x,y,z=L+f) = -\frac{i}{\lambda f} e^{ik(L+f)} e^{\frac{ik r^2}{2f}} e^{-i\frac{k}{2}\frac{r^2}{f^2}} FT(\text{aperture})$

with $k_x = \frac{kx}{f}$ $k_y = \frac{ky}{f}$

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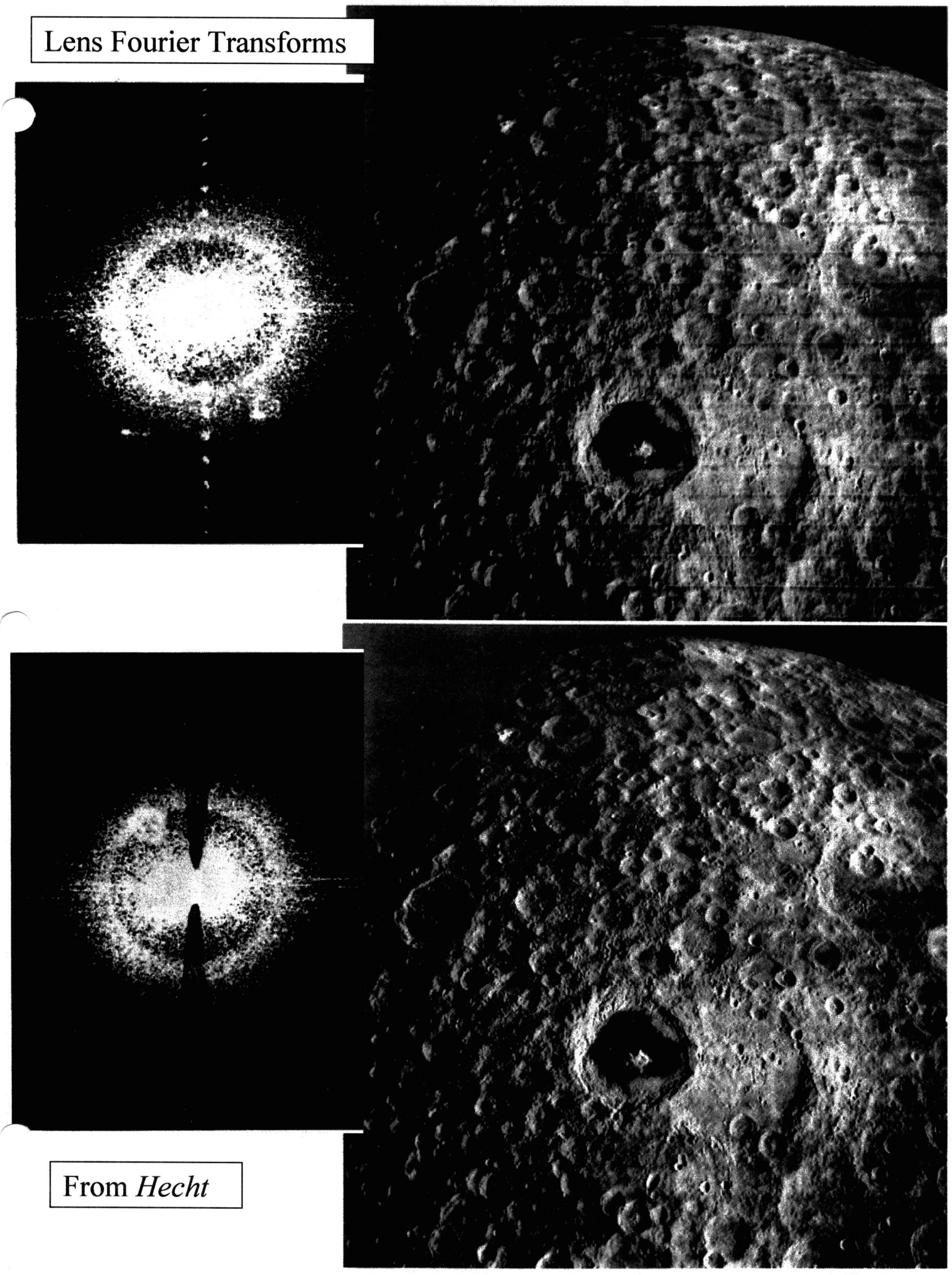
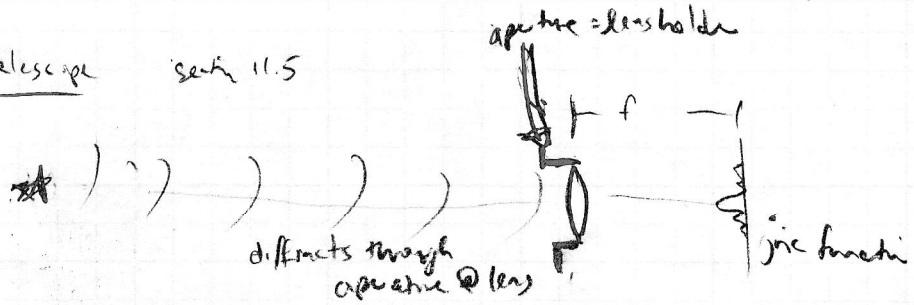
Lens Fourier Transforms



From *Hecht*

Telescope section 11.5



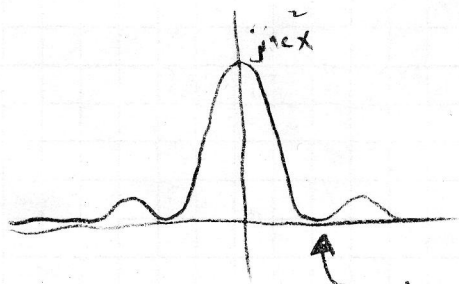
$$I = \frac{I_0}{\lambda^2 f^2}\left(FT(\text{circle})\right)^2$$

wrt $k_r = \frac{kr}{f}$ $r = \sqrt{x^2+y^2}$

$$(2\pi b^2)^2\left(\frac{J_1(k_r b)}{k_r b}\right)^2$$

$$I = \frac{I_0}{\lambda^2 f^2}(2\pi b^2)^2\left(\frac{J_1\left(\frac{k_r b}{f}\right)}{\frac{k_r b}{f}}\right)^2$$

b = radius of lens holder

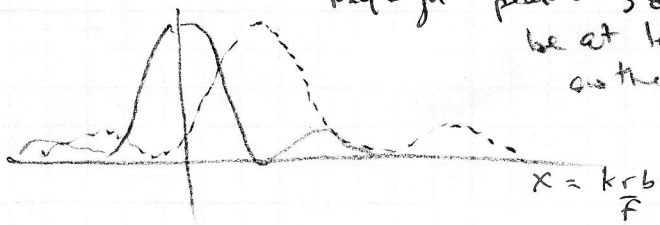


3.83171 ← 1.22 look familiar?

goes to zero at $= 1.21967\pi$

Resolve two stars?

Rayleigh: peak of star 2 needs to be at least as far away as the first minima



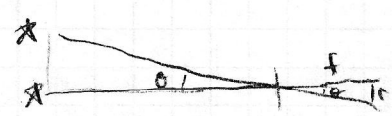
$$\frac{k_r b}{f} = 1.22\pi$$

l = diameter of lens

$$\frac{2\pi}{\lambda}\frac{r}{f}\left(\frac{l}{2}\right) = 1.22\pi$$

$$\boxed{\frac{r}{f} = \frac{1.22\lambda}{l}}$$

$$\boxed{\theta = \frac{1.22\lambda}{l}}$$



$\frac{r}{f} = \tan\theta \simeq \theta$

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Ch. 11 Many applications

Sect. 11.2 Diffraction of Gaussian field profile

laser beams typically brightest in middle, often have Gaussian fall-off

$$E(x',y') = E_0 e^{-\frac{r'^2}{w_0^2}} \qquad r' = \sqrt{x'^2+y'^2}$$

w_0 = "beam waist" (Scottish: not omega 😉)

measure of width → Radius at which Intensity $= I_0 \left(\frac{1}{e}\right)^2$

$= 13.5\%\ I_0$

How does it propagate? → diffraction through "infinitely large aperture"

Back to Fresnel approximation

$$E(x,y,z) = -\frac{i}{\lambda z} e^{ikz} e^{i\frac{k}{2z}(x^2+y^2)} \int_0^{2\pi}\int_0^{\infty} \left[E_0 e^{-r'^2/w_0^2}\right] e^{i\frac{k}{2z}r'^2} e^{-i\frac{k}{z}(xx'+yy')}\, r'\,dr'\,d\theta'$$

(r^2 on the first exponential; r'? on the second)

$xx' + yy' \to r\cos\theta\, r'\cos\theta'$, similar for y

$$-i\frac{k}{z}rr'(\cos\theta\cos\theta' - \sin\theta\sin\theta') \qquad \cos(\theta+\theta')$$

As before, θ integral $\to 2\pi J_0\left(\frac{krr'}{z}\right)$

$$= -\frac{i}{\lambda z} 2\pi e^{ikz} e^{i\frac{kr^2}{2z}} E_0 \int_0^{\infty} r'\,dr'\, e^{-\left(\frac{1}{w_0^2}-\frac{ik}{2z}\right)r'^2} J_0\left(\frac{krr'}{z}\right)$$

$\frac{2\pi}{\lambda} = k$

integral table $\int_0^{\infty} e^{-ax^2} J_0(bx)\, x\,dx = e^{-b^2/4a}/2a$

$$= \frac{e^{-\left(\frac{kr}{z}\right)^2 / 4\left(\frac{1}{w_0^2}-\frac{ik}{2z}\right)}}{2\left(\frac{1}{w_0^2}-\frac{ik}{2z}\right)}$$

← "rationalize" denominator

$= 2\frac{k}{2z}\left(\frac{2z}{kw_0^2} + i\right)$

put in polar form

use $\tan^{-1}\left(-\frac{1}{x}\right) = \tan^{-1}x - 90°$

introduce new symbols

$$E(x,y,z) = E_0 \frac{w_0}{w} e^{-r^2/w^2} e^{ikz + i\frac{kr^2}{2R} - i\tan^{-1}\left(\frac{z}{z_0}\right)}$$

with

$z_0 = \frac{kw_0^2}{2}$

$R = z + \frac{z_0^2}{z}$

$w = w_0\sqrt{1+\frac{z^2}{z_0^2}}$

} functions of z

△ incidentally I think you can show this satisfies Helmholtz eqn

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