Lecture 14: Fri, 8 Feb 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. The term describing a low-symmetry crystal is called

a. antisymmetric
b. homogeneous
c. non-isotropic
d. stratified

Q2. In a low-symmetry crystal, χ is represented by:

a. a real number
b. an imaginary number
c. a vector
d. a matrix (a tensor)

Q3. This very important equation determining the index of refraction along arbitrary angles in low-symmetry crystals,

$$\frac{u_x^{\,2}}{n^2 - n_x^{\,2}} + \frac{u_y^{\,2}}{n^2 - n_y^{\,2}} + \frac{u_z^{\,2}}{n^2 - n_z^{\,2}} = \frac{1}{n^2},$$ is called:

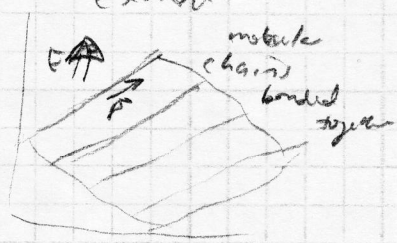
a. Fresnel's equation
b. Huygens' equation
c. Poynting's equation
d. Snell's equation

Crystals - ordered structure of atoms "non isotropic"
- could have preferred direction

$$\vec{P} = \epsilon_0 \overleftrightarrow{\chi} \vec{E}$$

$$\begin{pmatrix} E \end{pmatrix} = \epsilon_0 \underbrace{\begin{pmatrix} \chi \end{pmatrix}}_{\begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix}} \begin{pmatrix} E \end{pmatrix}$$

example



Simplicity: χ components = real (no absorption)

Simplicity: "ordinary crystals" χ = symmetric

then (matrix theorem) you can rotate coord. axes to get

i.e. "$\hat{x}$" = $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ in above picture

$$\begin{pmatrix} \chi_{xx} & & 0 \\ & \chi_{yy} & \\ 0 & & \chi_{zz} \end{pmatrix}$$ only

proved in Appendix 5.A

then $$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} \chi_{xx} & & \\ & \chi_{yy} & \\ & & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

gives:
$$\left.\begin{aligned} P_x &= \epsilon_0 \chi_{xx} E_x \\ P_y &= \epsilon_0 \chi_{yy} E_y \\ P_z &= \epsilon_0 \chi_{zz} E_z \end{aligned}\right\}$$ pretty similar to isotropic case

Simplicity (some crystals): $\chi_{xx} = \chi_{yy}$
χ_{zz}: different

↓

"uniaxial"
quartz, sapphire

Back to wave eqn (1.42), have no J_{free}

$$\nabla^2 E - \mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \cancel{\frac{\partial J_{free}}{\partial t}}^{=0} + \mu_0 \frac{\partial^2 P}{\partial t^2} - \frac{1}{\epsilon_0}\nabla(\nabla\cdot P)$$

Plane waves $\tilde{E} = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, similar for $\vec{B}$ and $\vec{P}$

Maxwell

$\nabla\cdot B = 0 \Rightarrow$ last time, $\vec{k}\cdot\vec{B} = 0$ $\qquad B \perp E$

$\nabla\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow$ last time $=0$ so $\vec{k}\cdot\vec{E} = 0$

now not! $\vec{D}$ not $\parallel \vec{E}$ so even if $\rho_{free}=0 \Rightarrow \nabla\cdot D = 0$ doesn't force $\nabla\cdot E = 0$

$\nabla\cdot D = 0$

instead $\Rightarrow \nabla\cdot(\epsilon_0 E + P) = 0$

$\vec{k}\cdot(\epsilon_0\vec{E} + \vec{P}) = 0$

if $\vec{P}$ not in $\vec{E}$ direction, then $\vec{k}\cdot\vec{E} \neq 0$

$\underline{\underline{E \text{ not } \perp k}}$! (but $\vec{D}$ is)

but $\vec{S} = \vec{E}\times\frac{\vec{B}}{\mu_0}$ must be $\perp \vec{E}$

so $\underline{\underline{\vec{S} \text{ not } \parallel k}}$

flow of energy + direction of wave propagation not the same!

Wave eqn:

$$-k^2 E - \mu_0\epsilon_0(-\omega^2 E) = \mu_0(-\omega^2)P - \frac{1}{\epsilon_0}\nabla(\overbrace{-\nabla\cdot\epsilon_0 E}^{\text{was } \nabla\cdot P})$$

check signs!

$$-k^2 E + \frac{\omega^2}{c^2}\mu_0(\epsilon_0 E + P) = +ik(ik\cdot E) \quad -k^2(k\cdot E)$$

$$k^2\vec{E} - \frac{\omega^2}{c^2}(\vec{E} + \overleftrightarrow{\chi}\vec{E}) = \vec{k}(k\cdot E)$$

$$k^2\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} - \frac{\omega^2}{c^2}\left[\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} + \begin{pmatrix}\chi_{xx}E_x\\\chi_{yy}E_y\\\chi_{zz}E_z\end{pmatrix}\right] = \begin{pmatrix}k_x\\k_y\\k_z\end{pmatrix}(\vec{k}\cdot\vec{E})$$

$$\left[k^2 - \frac{\omega^2}{c^2}(1+\chi_{xx})\right]E_x = k_x(\vec{k}\cdot\vec{E})$$

$$\left[k^2 - \frac{\omega^2}{c^2}(1+\chi_{yy})\right]E_y = k_y(\vec{k}\cdot\vec{E})$$

$$\left[k^2 - \frac{\omega^2}{c^2}(1+\chi_{zz})\right]E_z = k_z(\vec{k}\cdot\vec{E})$$

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$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \vec{k}\cdot\vec{E} \begin{pmatrix} k_x \Big/ k^2 - \frac{\omega^2}{c^2}(1+\chi_x) \\ k_y \Big/ k^2 - \frac{\omega^2}{c^2}(1+\chi_y) \\ k_z \Big/ k^2 - \frac{\omega^2}{c^2}(1+\chi_z) \end{pmatrix}$$

If eg $k^2 - \frac{\omega^2}{c^2}(1+\chi_x) = 0$ then mathematical problem ($\div 0$) here. We'll return to that case in example

multiply both sides by $(k_x \; k_y \; k_z)$

$$1 \quad \cancel{\vec{k}\cdot\vec{E}} = \cancel{\vec{k}\cdot\vec{E}} \left[\frac{k_x^2}{k^2 - \frac{\omega^2}{c^2}(1+\chi_x)} + \frac{k_y^2}{k^2 - \frac{\omega^2}{c^2}(1+\chi_y)} + \frac{k_z^2}{k^2 - \frac{\omega^2}{c^2}(1+\chi_z)} \right]$$

multiply both sides by $\frac{1}{c^2/\omega^2} \cdot \frac{1}{k^2}$

$$\frac{1}{k^2c^2/\omega^2} = \frac{u_x^2}{k^2c^2/\omega^2 - (1+\chi_x)} + \frac{u_y^2}{k^2\frac{c^2}{\omega^2} - (1+\chi_y)} + \frac{u_z^2}{k^2\frac{c^2}{\omega^2} - (1+\chi_z)}$$

with $\left.\begin{array}{l} u_x = k_x/k \\ u_y = k_y/k \\ u_z = k_z/k \end{array}\right\}$ $\vec{u}$ = unit vector in $\vec{k}$ direction

Substitute in $n^2 = \frac{k^2c^2}{\omega^2}$, def. $\left.\begin{array}{l} n_x = \sqrt{1+\chi_x} \\ n_y = \sqrt{1+\chi_y} \\ n_z = \sqrt{1+\chi_z} \end{array}\right\}$ as makes sense $n = \begin{pmatrix} n_x & & 0 \\ & n_y & \\ 0 & & n_z \end{pmatrix}$

$$\boxed{\frac{1}{n^2} = \frac{u_x^2}{n^2 - n_x^2} + \frac{u_y^2}{n^2 - n_y^2} + \frac{u_z^2}{n^2 - n_z^2}}$$

Fresnel's Eqn for crystal optics

Suppose you know direction of $\vec{k}$ (hence u_x, u_y, u_z)
and n_x, n_y, n_z (index of refraction along principal axes)

$\rightarrow$ then you can use this eqn to solve for n in your k-direction
(then you could get magnitude of k from $\frac{\omega}{k} = \frac{c}{n}$)

Awfully close to eqn for ellipsoid: multiply by n^2 and $\frac{1}{\frac{1}{n^2}}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$a = 1 - \left(\frac{n_x}{n}\right)^2 \quad \text{etc}$$

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Say $n = \begin{pmatrix} 1.5 & & 0 \\ & 1.6 & \\ 0 & & 1.7 \end{pmatrix}$ $\quad n_x = 1.5 \quad n_y = 1.6 \quad n_z = 1.7$

and $u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$$\frac{1}{n^2} = \frac{1}{n^2 - 1.5^2} + \frac{0}{n^2 - 1.6^2} + \frac{0}{n^2 - 1.7^2}$$

$$\frac{1}{n^2} = \frac{n^2}{n^2 - 1.5^2}$$

$$1 = \frac{n^2 - 1.5^2}{n^2}$$

$$1 = 1 - \frac{1.5^2}{n^2}$$

$$u = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$\left[\frac{1}{n^2} = \frac{1/2}{n^2 - 1.5^2} + \frac{1/2}{n^2 - 1.6^2} + 0 \right] \times n^2 (n^2 - 1.5^2)(n^2 - 1.6^2)$$

$$(n^2 - \overset{2.25}{1.5^2})(n^2 - \overset{2.56}{1.6^2}) = \tfrac{1}{2} n^2 (n^2 - 1.6^2) + \tfrac{1}{2} n^2 (n^2 - 1.5^2)$$

$$\cancel{n^4} - 4.81 n^2 + 5.76 = \cancel{.5 n^4} - 1.28 n^2 + \cancel{.5 n^4} - 1.125 n^2$$

$$5.76 = 2.405 n^2$$

$$n = \pm \sqrt{\frac{5.76}{2.405}} = \pm 1.54758\ldots$$

throw out negative sign

$$u = \frac{1}{\sqrt{3}}(1,1,1) \rightarrow n = 1.5391 \quad \text{(throw out negative sign)}$$

$$n = 1.65466$$

different n's for 2 polarizations

In general you get two n's (can solve numerically like ms, or use quadratic eqn for n^2 given in book

- Figuring out directions associated with the 2 n's is tougher. (Eigenvector vs eigenvalue)

special: $u = (1,0,0) \rightarrow n = 1.6$ or 1.7

but $\frac{0}{0}$ in eqns (can do it as limit)

Lecture 15: Mon, 11 Feb 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. For a uniaxial crystal, the index of refraction matrix (aka tensor) is given by:

$$\begin{pmatrix} n_x & & \\ & n_y & \\ & & n_z \end{pmatrix}$$

$$\begin{pmatrix} n_o & & \\ & n_o & \\ & & n_e \end{pmatrix}$$

a. $\begin{pmatrix} n_e & 0 & 0 \\ 0 & n_e & 0 \\ 0 & 0 & n_e \end{pmatrix}$

b. $\begin{pmatrix} n_0 & 0 & 0 \\ 0 & n_0 & 0 \\ 0 & 0 & n_0 \end{pmatrix}$

c. $\begin{pmatrix} n_0 & 0 & 0 \\ 0 & n_e & 0 \\ 0 & 0 & n_e \end{pmatrix}$

d. $\begin{pmatrix} n_0 & 0 & 0 \\ 0 & n_0 & 0 \\ 0 & 0 & n_e \end{pmatrix}$

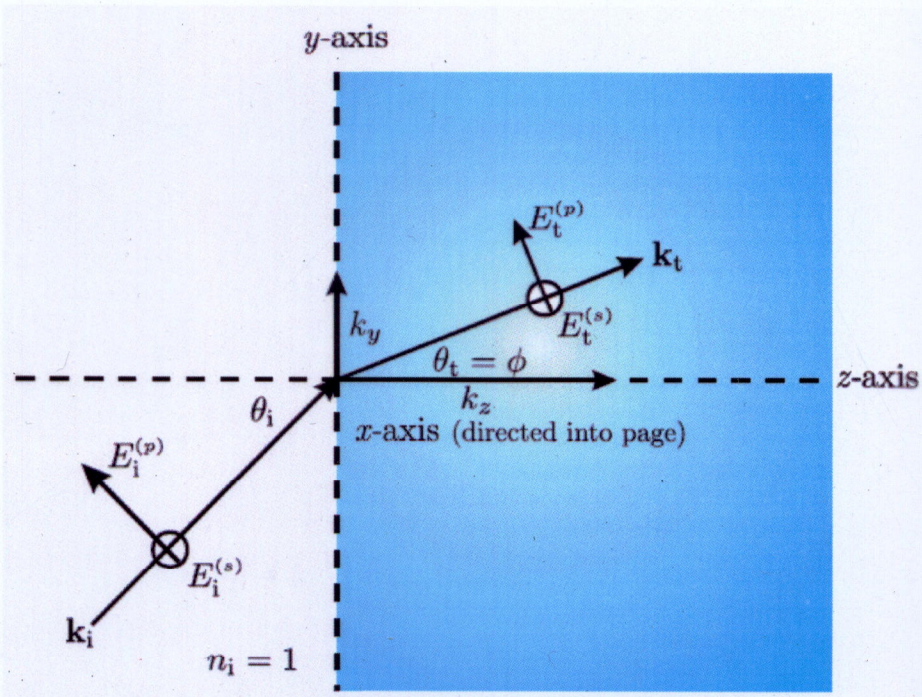
Q2. In a uniaxial crystal, which vector obeys Snell's law?

a. k

b. S

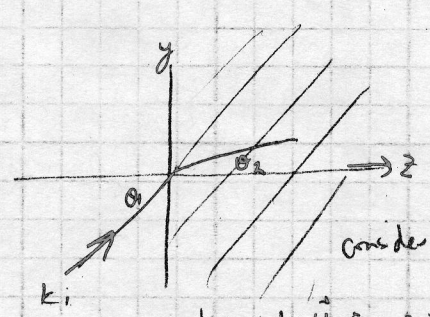
c. both k and S

Q3. T/F - S-polarized light entering a uniaxial crystal as shown (optic axis in z-direction) sees $n = n_o$ inside the crystal, regardless of incident angle.



<u>uniaxial</u> $n_x = n_o$
$n_y = n_o$
$n_z = n_e$ → unique direction, "optic axis"
like 3-fold symmetry axis of calcite.

Since x + y are indistinguishable, let's orient x-y axes so incident light is in y-z plane



consider crystal cut ⊥ to optic axis → opposite standard waveplate

inside crystal: $\hat{u} = \sin\theta_2\,\hat{y} + \cos\theta_2\,\hat{z}$

Fresnel eqn:

$$\frac{1}{n^2} = \frac{u_x^2}{n^2 - n_x^2} + \frac{u_y^2}{n^2 - n_y^2} + \frac{u_z^2}{n^2 - n_z^2}$$

becomes

$$\frac{1}{n^2} = \frac{0^2}{n^2 - n_o^2} + \frac{\sin^2\theta_2}{n^2 - n_o^2} + \frac{\cos^2\theta_2}{n^2 - n_e^2}$$

<algebra> solve for n in limit $u_x \to 0$

$$n = \begin{cases} n_o & \text{when } E \text{ is in x-direction (s polar)} \\ \dfrac{n_o\, n_e}{\sqrt{n_o^2 \sin^2\theta_2 + n_e^2 \cos^2\theta_2}} & \text{for p-polar extraordinary wave} \end{cases}$$

ordinary index doesn't vary w/ θ_2

θ_2 = angle from optic axis
important, call that "ϕ"

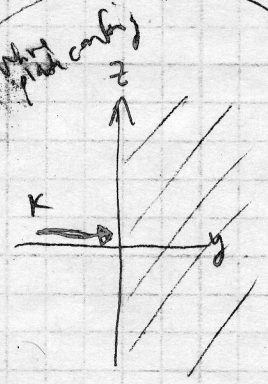
at $\phi = 0°$, get $n = \begin{cases} n_o \\ n_o \end{cases}$
makes sense w/

$$\frac{1}{n^2} = \frac{0^2}{n^2 - n_o^2} + \frac{0^2}{n^2 - n_o^2} + \frac{1^2}{n^2 - n_e^2}$$

n_o, n_o

maybe do this first?

Substrate: [illegible] plane cutting



$\phi = \frac{\pi}{2}$

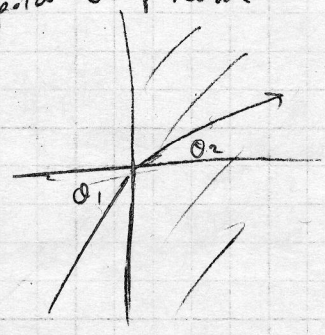
show + fast axes, n_o and n_e

$$\frac{1}{n^2} = \frac{0^2}{n^2 - n_o^2} + \frac{1^2}{n^2 - n_o^2} + \frac{0^2}{n^2 - n_e^2}$$

n_o, n_e

if $\vec{E}$ field in x-direction get n_x
if E field in z direction get n_z!

p-polar complication



θ_2 depends on n (Snell's Law)

n depends on θ_2 (via (u_x, u_y, u_z))

[s-polar not complicated, just use
$n_1 \sin\theta_1 = n_0 \sin\theta_2$]
$\theta_2 = \sin^{-1}\left(\frac{n_1}{n_0}\sin\theta_1\right)$

Need to solve simultaneously:

$$n_1 \sin\theta_1 = \left(\frac{n_0 n_e}{\sqrt{n_0^2 \sin^2\theta_2 + n_e^2 \cos^2\theta_2}}\right)\sin\theta_2$$

(algebra to solve for θ_2 as HW problem, w/ $n_1 = 1$)

$$\boxed{\theta_2 = \tan^{-1}\left(\frac{n_e \sin\theta_1}{n_0\sqrt{n_e^2 - \sin^2\theta_1}}\right)}$$

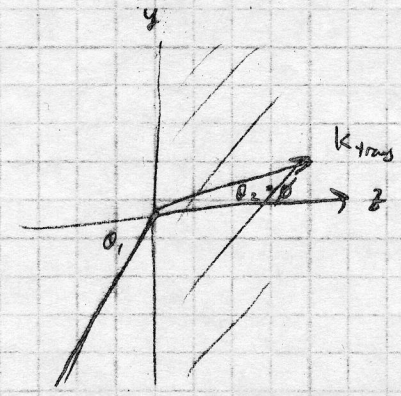
Snell's Law, with n_e function of θ_2!

"Extraordinary!" (Huygens)

Poynting vector

S polar : same as normal

P polar : weird!



k_{trans} direction given by $\tan\theta_2 = \dfrac{n_e \sin\theta_1}{n_o\sqrt{n_e^2 - \sin^2\theta_1}}$

$$\vec{S} = \frac{\vec{E}\times\vec{B}}{\mu_0}$$

What direction is E? $\nabla\cdot D = 0$. No phase = 0

$$\nabla\cdot(\epsilon_0 E + P) = 0$$

plane waves : $\vec{k}\cdot(\epsilon_0 E + P) = 0$ $\qquad P = \epsilon_0(\chi)E$

$$\vec{k}\cdot\epsilon_0\left[\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} + \begin{pmatrix}\chi_{xx}E_x\\ \chi_{yy}E_y\\ \chi_{zz}E_z\end{pmatrix}\right] = 0$$

$$\vec{k} = \begin{pmatrix}0\\ k\sin\phi\\ k\cos\phi\end{pmatrix}$$

$$k\sin\phi\,(1+\underbrace{\chi_{yy}}_{n_y^2 = n_o^2})E_y + k\cos\phi\,(1+\underbrace{\chi_{zz}}_{n_z^2 = n_e^2})E_z = 0$$

$$E_z = -\tan\phi\,\frac{n_o^2}{n_e^2}\,E_y$$

$$E = \begin{pmatrix}0\\ E_y\\ -\tan\phi\,\frac{n_o^2}{n_e^2}\,E_y\end{pmatrix}$$ field inside crystal

$$\vec{B} = \frac{1}{\omega}\,\vec{k}\times\vec{E}$$

$$= \frac{1}{\omega}\begin{pmatrix}0\\ k\sin\phi\\ k\cos\phi\end{pmatrix}\times\begin{pmatrix}0\\ E_y\\ -\tan\phi\,\frac{n_o^2}{n_e^2}E_y\end{pmatrix}$$

$$= \frac{1}{\omega}\left[-k\sin\phi\tan\phi\,\frac{n_o^2}{n_e^2}E_y - k\cos\phi\,E_y\right]\hat{x}$$

$$= -\frac{kE_y}{\omega}\hat{x}\left(\frac{n_o^2}{n_e^2}\sin\phi\tan\phi - \cos\phi\right)$$

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$$\vec{S} = \frac{-k}{\mu_0}\frac{E_y^2}{\omega}\begin{pmatrix} 0 \\ 1 \\ -\tan\phi\,\frac{n_o^2}{n_e^2} \end{pmatrix} \times \begin{pmatrix} \frac{n_o^2}{n_e^2}\sin\phi\tan\phi\cdot\cos\phi \\ 0 \\ 0 \end{pmatrix}$$

$$= +\frac{k}{\mu_0\omega}E_y^2\left(\frac{n_o^2}{n_e^2}\sin\phi\tan\phi\,\tan\phi\right)\underbrace{\left(\hat{z} + \hat{y}\,\frac{n_o^2}{n_e^2}\tan\phi\right)}_{\text{not in same direction as } \hat{k}\ (=\sin\phi\,\hat{y}+\cos\phi\,\hat{z})}$$

dir of $\vec{S}$ "ϕ'": $\tan\phi' = \dfrac{\frac{n_o^2}{n_e^2}\tan\phi}{1}$

$$\text{or } \tan\phi' = \frac{n_o^2}{n_e^2}\,\frac{\cancel{n_e}\sin\theta_1}{\cancel{n_o}\sqrt{n_e^2-\sin^2\theta_1}} = \frac{n_o}{n_e}\,\frac{\sin\theta_1}{\sqrt{n_e^2-\sin^2\theta_1}}$$

this is what is seen by an observer!

"extraordinary" beam doesn't appear to follow Snell's law, even after angular dependence of n_e is taken into account.

That's all for non isotropic crystals

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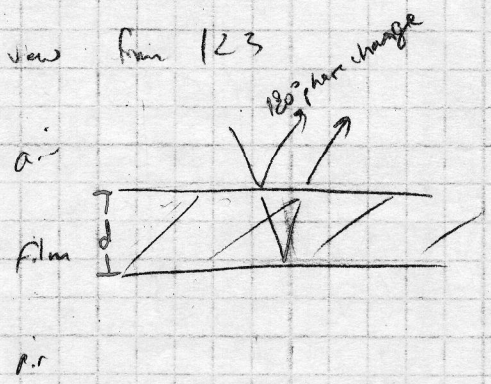
Chap 6: Multiple Parallel Interfaces

Review from 123



normal incidence: constructive interference

$$2d = \left(m + \tfrac{1}{2}\right) \lambda_{\text{in film}}$$

$\lambda_{\text{in film}} = \lambda/n$

to match 180° phase shift on surface

i.e. "optical path length" $= \left(m + \tfrac{1}{2}\right) \lambda$

$$OPL = (P.L.) \times n$$

where does OPL come from in our eqns?

$$e^{i(\vec{k}\cdot\vec{r} - \omega t)}$$

phase shift between two pts at same time

$$= e^{i \vec{k}\cdot(\Delta\vec{r})}$$

eg between 0 and $d\hat{z}$

$$= e^{ikd}$$

$$= e^{i 2\pi \frac{1}{\lambda} n d}$$

$$= e^{i 2\pi n d/\lambda}$$

$$\frac{\omega}{k} = \frac{c}{n}$$

$$k = \frac{\omega}{c} \cdot n$$

$$= 2\pi \frac{f}{c} n$$

$$= 2\pi \frac{1}{\lambda} n$$

when nd/λ = integer

have constructive interference

(or destructive, if 180° phase shift)