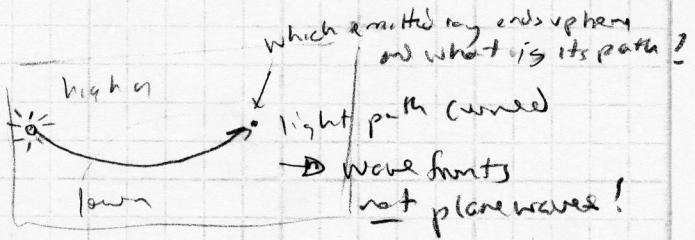


day 28

- Bret to give this lecture -

## Chap 9: Light as rays

Suppose  $n$  changes with position.

Try to solve wave eqn

$$\nabla^2 E = \frac{1}{r^2} \frac{\partial^2 E}{\partial r^2}$$

$$\nabla^2 E - \frac{n^2(\vec{r})}{c^2} \frac{\partial^2 E}{\partial t^2} = 0$$

$$\text{Trial soln: instead of } \vec{E} = \vec{E}_0 e^{i(k \cdot \vec{r} - \omega t)}$$

$$\vec{E} = \vec{E}_0(\vec{r}) e^{i(k_{vac} R(\vec{r}) - \omega t)}$$

ie if  $n$  not varying,  
 $R = \frac{k \cdot \vec{r}}{k_{vac}}$

(=  $\vec{z}$  in vacuum  
w/plane waves  $\vec{r} \approx \vec{z}$ )

$$\frac{\partial^2}{\partial r^2} \rightarrow (-i\omega)$$

$$\frac{\partial^2}{\partial t^2} \rightarrow -\omega^2$$

then we have  $\frac{\omega^2}{c^2} = k_{vac}^2$  in right hand term  
divide by this

$$\frac{1}{k_{vac}} \nabla^2 (\vec{E}_0(\vec{r}) e^{i k_{vac} R(\vec{r}) - i \omega t}) + n^2(\vec{r}) \vec{E}_0(\vec{r}) e^{i k_{vac} R(\vec{r}) - i \omega t} = 0$$

HW problem: what is  $\nabla^2 (\vec{E}_0(\vec{r}) e^{i k_{vac} R(\vec{r})})$ ?

use defn  $\nabla^2 f = \nabla \cdot (\nabla f)$  for each component

$$\text{for } E_{ox}: = \nabla \cdot \left[ E_{ox}(\vec{r}) \nabla (e^{i k_{vac} R(\vec{r})}) + e^{i k_{vac} R(\vec{r})} \nabla (E_{ox}(\vec{r})) \right] \quad \begin{matrix} \text{using} \\ \text{product} \\ \text{rule} \end{matrix}$$

- use more P.R.'s, get factors of  $\nabla(R)$  from chain rule

combine  $E_x, E_y, E_z$  results  
plug back into wave eqn

$$[\vec{\nabla}R \cdot \vec{\nabla}R - n^2] E_0(\vec{r}) = \frac{\nabla^2 \vec{E}_0(\vec{r})}{k_{\text{vac}}^2} + i \frac{\nabla^2 R}{k_{\text{vac}}} + \frac{2i}{k_{\text{vac}}} (\nabla E_0 \cdot \nabla R) + \hat{y} + \hat{z} \text{ terms}$$

**Big Approx:** short wavelength  $\lambda_{\text{vac}} \rightarrow 0$   $\frac{1}{k_{\text{vac}}} \rightarrow 0$  } only LHS survives

Rays: commonly describe features that are large compared to a wavelength

$$\vec{\nabla}R(\vec{r}) \cdot \vec{\nabla}R(\vec{r}) = n^2(\vec{r})$$

$$(\nabla R)^2 = n^2$$

$$\text{mag. of } \nabla R = n$$

$$\nabla R(\vec{r}) = n(\vec{r}) \hat{s} \quad \text{Eikonal Eqn}$$

$\hat{s}$  can change with  $\vec{r}$  also!

HW 9.3 Pointing vector  $\rightarrow$  DR direction  $\rightarrow \hat{s} = \text{dir. of energy flow}$   
at each pt in space

Remember Fermat's principle?  $\underbrace{\text{path} =}_{\text{(least time)}}$

$$\text{proof: } \underbrace{\nabla \times (\nabla R)}_{=0} = \nabla \times (n \hat{s})$$

curl of gradient

$$\text{Integrate over surface: } \int_S \nabla \times (n \hat{s}) d\alpha = 0$$

$$\oint_P (n \hat{s}) \cdot d\vec{l} = 0 \quad (\text{Stokes Thm})$$

Recall  
scalar potential functi  
 $\oint \vec{E} \cdot d\vec{l} = 0$   
 $\rightarrow V = - \int \vec{E} \cdot d\vec{l}$

corollary:  $\int_A^B (n \hat{s}) \cdot d\vec{l} = \text{path independent}$  keep in mind both  $n$  and  $\hat{s}$  are changing w/ position



pick one path from B to A

then all other return paths have to give  $(-\text{the integral})$  for some things  
So that  $\oint = 0$

$$\int_A^B n \cos \theta d\ell = \text{path independent}$$

$\cos \theta = \text{angle between } \hat{s} \text{ and } d\vec{l}$

How does this compare to  $\int_A^B n \cdot dl$ ? (It's always  $\leq$ , since  $\cos \theta$  always  $\leq 1$ )

$$\boxed{\int_A^B n \cdot dl = OPL}$$

(should look familiar)

~ before  $n \cdot dl = OPL$

→ Figure out which of all possible paths has the minimal OPL, then  $\int_A^B n \cdot dl$  must still be  $\leq$  to this

$$\left| \int_A^B (n \hat{s}) \cdot d\hat{s} \right| = \text{smallest possible OPL}$$

(=, obviously)

antipath

therefore  
it really is in  
direction of  
actual path

$= c \times \text{smallest possible time!}$

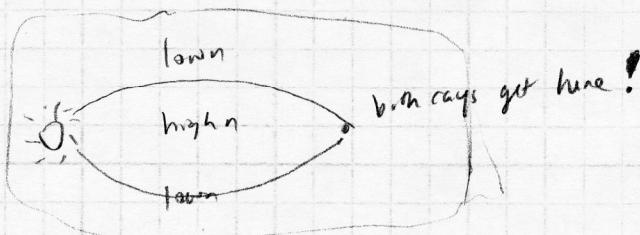
since  $t = \frac{\text{distance}}{\text{velocity}}$

$$\text{time for path} = \int \frac{dl}{c/n(r)} \quad \text{if } n \text{ changes}$$

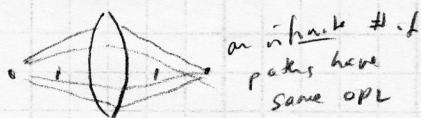
$$= \frac{1}{c} \times OPL$$

Note: doesn't apply to crystals where  $n$  depends on direction, not just position

What if more than one path has same OPL?



Common example: a lens!



on infinitely # of paths have same OPL

day 29 - Bret to give this lecture too -

## ABCD Matrices

- characterize light ray at a particular spot with two parameters

$$\underline{\underline{X}}_1 = \begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix} \quad y_1, \alpha_1 \rightarrow \text{axis of optical system}$$

"Paraxial":  $y$  and  $\alpha$  are small (close to axis)

at another point, that ray will be different already

$$\underline{\underline{X}}_2 = \begin{pmatrix} y_2 \\ \alpha_2 \end{pmatrix}$$

Represent change via a  $2 \times 2$  matrix!

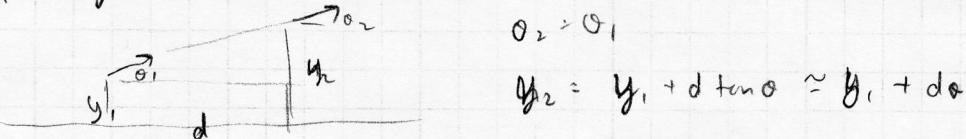
$$\underline{\underline{X}}_2 = M \underline{\underline{X}}_1$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = 4 \text{ components of } M$$

$$\det M = AD - BC$$

"Beauty": Effects of several situations  $\rightarrow$  just multiply matrices together!

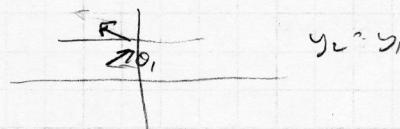
Situation 1: straight line motion, distance  $d$



$$\begin{pmatrix} y_2 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \alpha_1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$

Situation 2: Reflection from flat mirror



what angle to use for  $\theta_2$ ?

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

$\hookrightarrow$  trivial M

conversion:  $\overleftarrow{\theta_2}$  if going left

then  $\theta_2 = -\theta_1$

Situation 3: Snell's law  
Refraction from flat surface



$$n_1 \sin \theta_1 = n_2 \sin \theta_2$$

$$n_1 \theta_1 \approx n_2 \theta_2$$

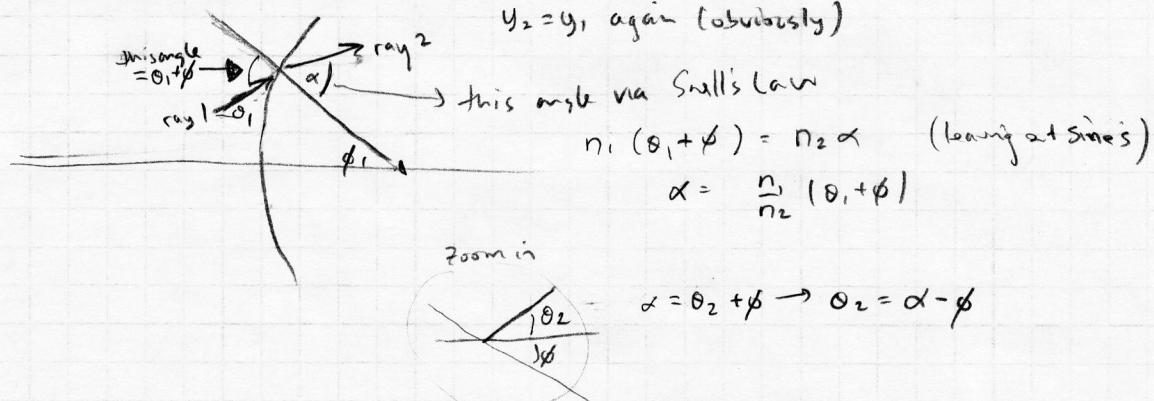
$$\theta_2 = \frac{n_1}{n_2} \theta_1$$

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

Situation 4: Snell's law refraction from curved surface (like a lens)

R = radius of curvature

$y_2 = y_1$  again (obviously)



$$\begin{aligned} \theta_2 &= \frac{n_1}{n_2} \theta_1 + \frac{n_1}{n_2} \phi - \phi \\ &= \frac{n_1}{n_2} \theta + \left( \frac{n_1}{n_2} - 1 \right) \phi \end{aligned}$$

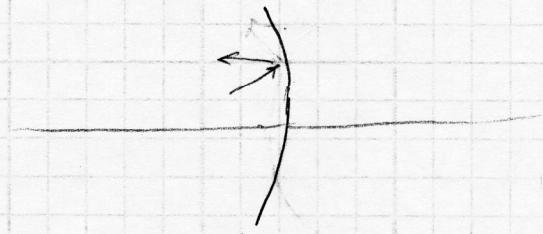
What is  $\phi$  in terms of  $y$ ?  $\phi = \frac{y}{R}$

$$\theta_2 = \frac{n_1}{n_2} \theta + \left( \frac{n_1}{n_2} - 1 \right) \frac{1}{R} y$$

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \frac{1}{R} \left( \frac{n_1}{n_2} - 1 \right) & \frac{n_1}{n_2} \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

Notice this M reduces to our previous M when  $R = \infty$   
Note: opposite curvature handled via negative R

## Situation 5 (almost done!) Reflection from curved surface



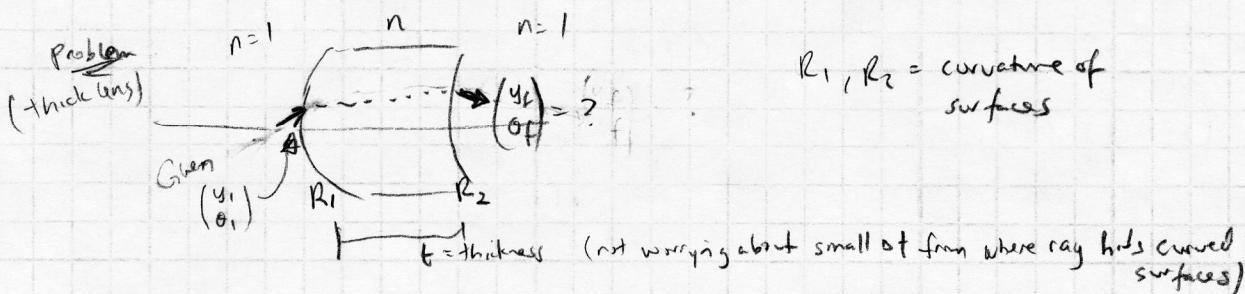
skip work

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix} \quad \text{with } f = \frac{R}{2}$$

(opposite curvature  $\rightarrow$  negative  $f$ )

We now have 3 very important matrices that we can piece together for more complicated situations

$\downarrow$   
because the two flat ones are subsets  
( $R \rightarrow \infty$ )



maybe use  
d instead  
of t

$$\begin{pmatrix} y_f \\ \theta_f \end{pmatrix} = (\text{surface 2})(\text{translation})(\text{surface 1}) \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

$$= \underbrace{\left( \begin{array}{cc} 1 & 0 \\ \frac{1}{R_2}(\frac{n}{n-1}) & \frac{n}{n-1} \end{array} \right)}_{t=\text{thickness}} \left( \begin{array}{cc} 1 & t \\ 0 & 1 \end{array} \right) \underbrace{\left( \begin{array}{cc} 1 & 0 \\ \frac{1}{R_1}(\frac{n}{n-1}) & \frac{n}{n-1} \end{array} \right)}_{t=\text{thickness}} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

$$M_{\text{tot}} = \begin{pmatrix} 1 - \frac{t}{R_1}(t - \frac{1}{n}) & \frac{t}{n} \\ \frac{n-1}{R_2} - (1 - \frac{t}{n})(n + \frac{t}{R_2}(n-1)) & 1 + \frac{t}{R_2}(t - \frac{1}{n}) \end{pmatrix} \quad \text{done w/ Mathematica}$$

The answer!

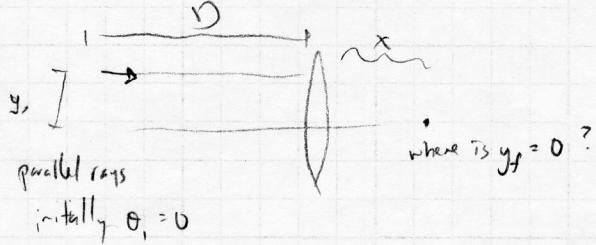
Interesting:  $t \rightarrow 0$  (thin lens)

$$M_{\text{tot}} = \begin{pmatrix} 1 & 0 \\ (n-1)(\frac{1}{R_2} - \frac{1}{R_1}) & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} M!$$

if outside =  $n_1$   
inside =  $n_2$  then  $n \rightarrow \frac{n_2}{n_1}$

with  $\frac{1}{f} = (n-1)(\frac{1}{R_1} - \frac{1}{R_2})$

What does a matrix of form  $\begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix}$  imply?



where is  $y_f = 0$ ?

$$\begin{pmatrix} y_f \\ 0_f \end{pmatrix} = (\text{transl. of } x) \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} (\text{transl. of } l) \begin{pmatrix} y_i \\ 0_i \end{pmatrix}$$

*thin lens  
or focusing mirror*

$$\begin{pmatrix} 0 \\ 0_f \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\frac{1}{f} & 1 \end{pmatrix} \begin{pmatrix} 1 & ? \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_i \\ 0 \end{pmatrix}$$

$(y_i)$

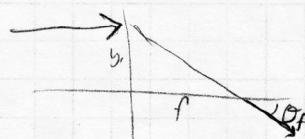
$\begin{pmatrix} y_i \\ -y_i/f \end{pmatrix}$

$$\begin{pmatrix} y_i - x \frac{y_i}{f} \\ -\frac{y_i}{f} \end{pmatrix}$$

top row:  $0 = y_i - x \frac{y_i}{f} \Rightarrow \boxed{x = f}$

All parallel rays converge to axis at  $x=f$   
regardless of initial height!

They converge with angle of  $-\frac{y_i}{f}$



makes perfect sense!