

## Lecture 3: Fri, 11 Jan 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. T/F - Maxwell's main contribution to "Maxwell's Equations" (aside from organizing other people's formulas) was to modify Ampere's Law so it became applicable to dynamic situations.

Q2. T/F - Faraday's Law is a special case of Ampere's Law.

Q3. The "continuity equation", also known as the "equation of continuity" is which of the following:

a.  $\vec{\nabla} \cdot \vec{B} = 0$

b.  $\vec{\nabla} \times \vec{B} = \mu_0 J$

c.  $\vec{\nabla} \times \vec{B} = \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

d.  $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

e.  $\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$

3. No magnetic monopoles → 'Gauss's Law for  $\vec{B}$ '

$$\oint \vec{B} \cdot d\vec{a} = 0$$

$$\int_V (\nabla \cdot \vec{B}) dV = 0$$

true for all volumes

$$\rightarrow \boxed{\nabla \cdot \vec{B} = 0}$$

4. Biot Savart Law, force between a current & a moving charge

$$\vec{F} = q \vec{v} \times \vec{B}$$



$$\vec{B} = \frac{\mu_0}{4\pi} \int \frac{d\vec{I} \times \hat{r}}{r^2} \quad (dI = Id\vec{l}) \quad \text{true for "static currents"}$$

$$\boxed{\vec{B} = \frac{\mu_0}{4\pi} \int_V \frac{\vec{J} \times \hat{r}}{r^2} dV'}$$

$\vec{J}$  = current density,  $\frac{\text{current}}{\text{area}}$

from that, we get

Ampere's Law

$$\oint \vec{B} \cdot d\vec{l} = \mu_0 I_{enc} \quad (\text{if } J \text{ is not changing})$$

↓  
Stoke's Theorem

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \int_S \mu_0 \vec{J} \cdot d\vec{a}$$

true for all surfaces

$$\rightarrow \underline{\underline{\nabla \times \vec{B} = \mu_0 \vec{J}}} \quad \text{true for statics}$$

day 3

Colton: if a changing B field can produce an E field (Faraday's Law) shouldn't a changing E field produce a B field?

Yes → something's missing

Review:

$$\begin{aligned} \nabla \cdot \vec{E} &= \rho/\epsilon_0 \\ \nabla \times \vec{E} &= -\dot{\vec{B}}/\epsilon_0 \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{B} &= \mu_0 \vec{J} + \dots \end{aligned}$$

# Maxwell's Fix to Ampere's Law

Note there's a fundamental mathematic problem

vector math: div of a curl = 0 always. HW P 0.16

$$\nabla \cdot (\nabla \times \vec{E}) \text{ must } = 0$$

$$\text{test it out: } = \nabla \cdot \left( \nabla \times \frac{d\vec{B}}{dt} \right)$$

$$= -\frac{1}{dt} (\nabla \cdot \vec{B})$$

$$= 0 \checkmark$$

what about  $\nabla \cdot (\nabla \times \vec{B})$ ?

$$\nabla \cdot (\nabla \times \vec{B}) = \nabla \cdot (\mu_0 \vec{J} + ?)$$

$$= \mu_0 (\nabla \cdot \vec{J}) + \nabla \cdot (?)$$

↓  
Not zero!

Conservation of charge: if charge flows out,  $\rho$  decreases



$$Q \text{ inside} = \int \rho \, dV$$

$$I \text{ flows out} = \int \vec{J} \cdot d\vec{a} = -\frac{dQ}{dt}$$

(negative because direction J reverses Q)

$$\int \vec{J} \cdot d\vec{a} = -\int \frac{d\rho}{dt} \, dV$$

$$\int (\nabla \cdot \vec{J}) \, dV = -\int \left( \frac{d\rho}{dt} \right) \, dV$$

true for all V

$$\boxed{\nabla \cdot \vec{J} = -\frac{d\rho}{dt}}$$

Eqn of continuity

$$0 = \mu_0 \left( -\frac{d\rho}{dt} \right) + \nabla \cdot (?)$$

$$\rho = \epsilon_0 \nabla \cdot \vec{E} \text{ by ME \#1}$$

$$0 = \mu_0 \epsilon_0 \nabla \cdot \left( \frac{d\vec{E}}{dt} \right) + \nabla \cdot (?)$$

$$? = \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}$$

$$\boxed{\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{d\vec{E}}{dt}}$$

→ related to "displacement current"  
 $\frac{d\vec{E}}{dt}$  → seen in eg charging capacitor

→ Ampere's Law Song

physics songs.org

integral form  $\int_C (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 I + \mu_0 \epsilon_0 \frac{d}{dt} \int_S \vec{E} \cdot d\vec{a}$

(circulate) (flux)

"he saw that light could move through empty space"  
 "a changing B field makes an E field and vice versa. all at the perfect pace"

Wave eqn

Start w/

$$\begin{cases} \nabla \cdot E = \rho/\epsilon_0 \\ \nabla \times E = -\frac{dB}{dt} \\ \nabla \cdot B = 0 \\ \nabla \times B = \mu_0 J + \mu_0 \epsilon_0 \frac{dE}{dt} \end{cases}$$

Consider a vacuum.  $\rho=0, J=0$

- Then
1.  $\nabla \cdot E = 0$
  2.  $\nabla \times E = -\frac{dB}{dt}$
  3.  $\nabla \cdot B = 0$
  4.  $\nabla \times B = \mu_0 \epsilon_0 \frac{dE}{dt}$

Vector identity: HW Problem P. 17  $\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$

gradient of this vector field  $\nabla^2 A = \frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} + \frac{\partial^2 A}{\partial z^2}$  Laplacian

Take 2<sup>nd</sup> Maxwell Eqn

$$\nabla \times (\nabla \times E) = \nabla \times \left( -\frac{dB}{dt} \right)$$

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{d}{dt} (\nabla \times B)$$

$\downarrow 0$   $\mu_0 \epsilon_0 \frac{dE}{dt}$

$\nabla^2 E = \mu_0 \epsilon_0 \frac{d^2 E}{dt^2}$  "the wave equation" with  $\mu_0 \epsilon_0 = \frac{1}{v^2}$

Similarly  $\nabla \times (\nabla \times B) \rightarrow \nabla^2 B = \mu_0 \epsilon_0 \frac{d^2 B}{dt^2}$

$v = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$   
 $\mu_0 = 4\pi \cdot 10^{-7}$   
 $\epsilon_0 = 8.854 \cdot 10^{-12}$   
 $\rightarrow v = 3 \cdot 10^8 \text{ m/s} !!!$

Why called wave eqn?

Consider eqn  $\frac{d^2 f}{dx^2} = \frac{1}{v^2} \frac{d^2 f}{dt^2}$  "1D wave eqn"

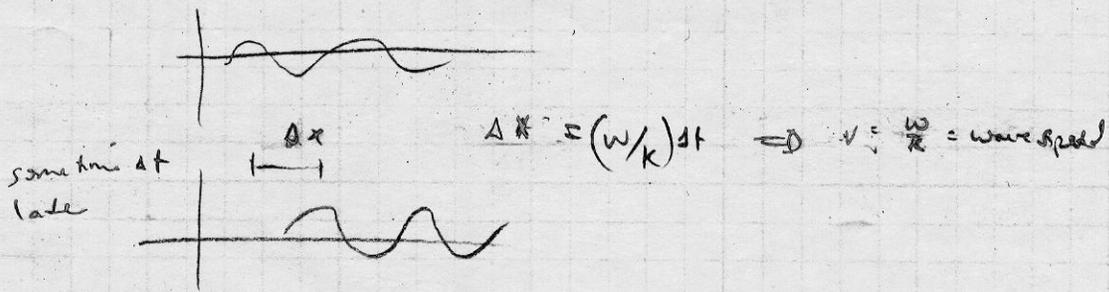
Solutions are travelling waves like  $f = A \sin(kx - \omega t)$

$$\frac{d^2 f}{dx^2} = -A k^2 \sin(kx - \omega t) = -k^2 f$$

$$\frac{d^2 f}{dt^2} = -A \omega^2 \sin(kx - \omega t) = -\omega^2 f$$

divide  $\frac{d^2 f/dx^2}{d^2 f/dt^2} = \frac{k^2}{\omega^2}$

$\therefore \frac{d^2 f}{dx^2} = \frac{1}{(v/k)^2} \frac{d^2 f}{dt^2} \quad v = \frac{\omega}{k}$



$\frac{d^2}{dx^2} \rightarrow \nabla^2$  then you have 3D wave eqn!

Fourier: any (finite) wave can be made up of sines + cosines

$\rightarrow$  any function of the form  $f(kx - \omega t)$   
or  $f(x - vt)$  } is a soln

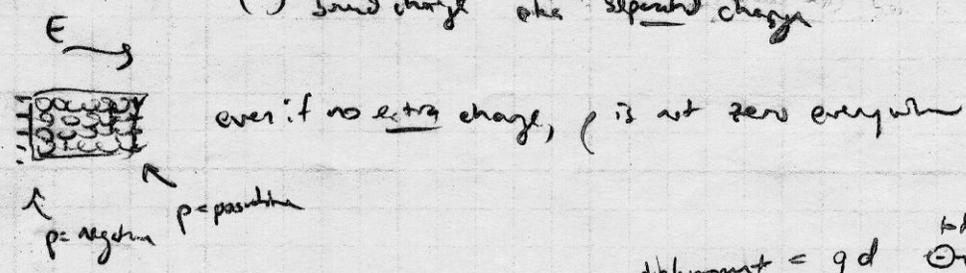
g.  $e^{i(kx - \omega t)}$

Maxwell eqns in matter

what about regions inside matter (dielectric) must modify M.E. (at least it's useful to do so)

$\nabla \cdot E = \rho_{free}$

- $\rho$  from two sources  
 (1) "free charge" aka extra charge  
 (2) "bound charge" aka "separated charge"



define  $\vec{P}$  = dipole moment / volume

dipole moment =  $q d$  for an isolated molecule

then  $\rho_b = -\nabla \cdot \vec{P}$  (turns out proved in Griffiths)

$\nabla \cdot \vec{E} = \frac{\rho_{free} + \rho_{bound}}{\epsilon_0} = \frac{\rho_{free}}{\epsilon_0} + \frac{-\nabla \cdot \vec{P}}{\epsilon_0}$

$\nabla \cdot (\vec{E} + \frac{1}{\epsilon_0} \vec{P}) = \frac{\rho_{free}}{\epsilon_0}$

$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_{free}$

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$  = a mathematically useful field "displacement field"

$\nabla \cdot \vec{D} = \rho_{free}$

ME #1 for materials  
 "Macroscopic ME"  
 ↳ don't need to worry about microscopic details of  $P$ , just the  $P$  that's easy to observe

$\nabla \times B = \mu_0 J + \mu_0 \frac{dE}{dt}$

- $J$  from 3 sources  
 (1) "free current" aka macroscopic current  
 (2) "polarized current" from "shaking" of material as it's becoming polarized

$\vec{J}_p = \frac{d\vec{P}}{dt}$  (proved in Griffiths)

(3) bound current

**Lecture 4: Mon, 14 Jan 2008**

Reading quizzes: no talking, no looking in your books/notes

Q1. Complete the four fundamental (“microscopic”) Maxwell equations:

$$\vec{\nabla} \cdot \vec{E} = \dots$$

$$\vec{\nabla} \times \vec{E} = \dots$$

$$\vec{\nabla} \cdot \vec{B} = \dots$$

$$\vec{\nabla} \times \vec{B} = \dots$$

Q2. Which two Maxwell’s equations must be modified to describe electric & magnetic fields inside materials (to get the “macroscopic” equations)?

a.  $\vec{\nabla} \cdot \vec{E}$  and  $\vec{\nabla} \times \vec{E}$

b.  $\vec{\nabla} \cdot \vec{B}$  and  $\vec{\nabla} \times \vec{B}$

c.  $\vec{\nabla} \cdot \vec{E}$  and  $\vec{\nabla} \times \vec{B}$

d.  $\vec{\nabla} \times \vec{E}$  and  $\vec{\nabla} \cdot \vec{B}$

Q3. The wave equation given in the book for general use (i.e. inside materials not just in vacuum) is the wave equation in a vacuum, plus how many additional “complicated source terms”?

a. 1

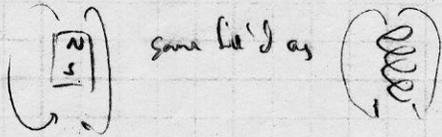
b. 2

c. 3

d. 4

e. 5

bound current: from intrinsically magnetic objects



same field as

day 4

must be "currents" inside!

QM: spin of electrons = intrinsic angular momentum

also potential orbital ang. momentum

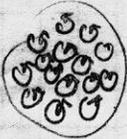


Giffiths Fig 6.15

turns out  $\vec{J}_b = \nabla \times \vec{M}$

define  $M = \frac{\text{mag. dipole moment}}{\text{volume}}$

mag. dip. mom =  $I \times \text{area}$   
for wire loop



$$\begin{aligned} \text{then } \nabla \times \vec{B} &= \mu_0 \vec{J}_{\text{free}} + \mu_0 \vec{J}_p + \mu_0 \vec{J}_b + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ &= \mu_0 \vec{J}_{\text{free}} + \mu_0 \left( \frac{\partial \vec{P}}{\partial t} \right) + \mu_0 (\nabla \times \vec{M}) + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \end{aligned}$$

$$\nabla \times \vec{B} - \nabla \times \mu_0 \vec{M} = \mu_0 \vec{J}_p + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$$

divide by  $\mu_0$ ,

$$\nabla \times \left( \frac{\vec{B}}{\mu_0} - \vec{M} \right) = \vec{J}_p + \frac{\partial}{\partial t} (\vec{P} + \epsilon_0 \vec{E})$$

$\underbrace{\qquad\qquad\qquad}_{\text{D already}}$

define  $\vec{H} = \frac{\vec{B}}{\mu_0} - \vec{M}$

$$\nabla \times \vec{H} = \vec{J}_{\text{free}} + \frac{\partial \vec{D}}{\partial t}$$

↳ "displacement current"

ME in matter:

$$\begin{aligned} \nabla \cdot \vec{D} &= \rho_f \\ \nabla \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ \nabla \cdot \vec{B} &= 0 \\ \nabla \times \vec{H} &= \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \end{aligned}$$

"constitutive eqns"

$$\begin{aligned} \vec{D} &= \epsilon_0 \vec{E} + \vec{P} \\ \vec{H} &= \frac{\vec{B}}{\mu_0} - \vec{M} \end{aligned}$$

permeability

$$= \epsilon_0 \epsilon_r \vec{E} \text{ after } (\vec{P} \propto \vec{E})$$

permeability

$$= \frac{1}{\mu_0 \mu_r} \vec{B} \text{ after } (\vec{M} \propto \vec{B})$$

relative permeability

$\epsilon_r = \text{relative permittivity} = \text{dielectric constant} (K?)$

Wave Eqn inside dielectric/magnetic material

→ some  $\rho_b$ , no  $\rho_f$   
 some  $J_b$ , no  $J_f$

1.  $\nabla \cdot D = \rho_f$
2.  $\nabla \times E = -\partial B / \partial t$
3.  $\nabla \cdot B = 0$
4.  $\nabla \times H = J_f + \partial D / \partial t$

$$\Rightarrow \begin{aligned} \nabla \cdot D &= 0 \\ \nabla \times E &= -\partial B / \partial t \\ \nabla \cdot B &= 0 \\ \nabla \times H &= \partial D / \partial t \end{aligned}$$

if linear,  $D = \epsilon_0 \epsilon_r E$   
 $H = \frac{1}{\mu_0 \mu_r} B$

$$\Rightarrow \begin{aligned} \nabla \cdot E &= 0 \\ \nabla \times E &= -\partial B / \partial t \\ \nabla \cdot B &= 0 \\ \nabla \times \left( \frac{1}{\mu_r} B \right) &= \epsilon_0 \epsilon_r \frac{\partial E}{\partial t} \end{aligned}$$

$\nabla \times B = \mu_0 \epsilon_r \mu_r \frac{\partial E}{\partial t}$   
 ↓  
 only change

get wave eqn again

$$\begin{aligned} \nabla^2 E &= \mu_0 \epsilon_0 \mu_r \epsilon_r \frac{\partial^2 E}{\partial t^2} \\ \nabla^2 B &= \mu_0 \epsilon_0 \mu_r \epsilon_r \frac{\partial^2 B}{\partial t^2} \end{aligned}$$

$$v = \frac{1}{\sqrt{\mu_r \epsilon_r}} \frac{1}{\sqrt{\mu_0 \epsilon_0}}$$

$$v = \frac{c}{\sqrt{\epsilon_r \mu_r}}$$

compare  $v = \frac{c}{n}$  learned somewhere

if  $\mu_r \approx 1$ , as it often is, then  $n = \sqrt{\epsilon_r}$

index of refraction =  $\sqrt{\text{dielectric constant}}$

To get P+W's wave eqn.

start with  $\nabla \cdot D = \rho_{free}$

$$D = \epsilon_0 E + P$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad \text{Max}$$

$$\nabla \cdot B = 0 \quad \text{Max}$$

$$\nabla \times H = J_{free} + \frac{dD}{dt}$$

$$H = \frac{B}{\mu_0} - M$$

Restrict to materials where  $\rho_{free} = 0$   
 $M = 0$

Then  $\nabla \cdot (\epsilon_0 E + P) = 0 \rightarrow \nabla \cdot E = -\frac{1}{\epsilon_0} \nabla \cdot P$

$$\nabla \times \left( \frac{B}{\mu_0} \right) = J_{free} + \frac{d}{dt} (\epsilon_0 E + P)$$

$$\rightarrow \nabla \times B = \mu_0 J_{free} + \mu_0 \epsilon_0 \frac{dE}{dt} + \mu_0 \frac{dP}{dt}$$

Then do the usual  $\nabla \times (\nabla \times E) = \nabla \times \left( -\frac{\partial B}{\partial t} \right)$  trick

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{\partial}{\partial t} (\nabla \times B)$$

$$\nabla \left( \frac{1}{\epsilon_0} \nabla \cdot P \right)$$

$$\mu_0 J_{free} + \mu_0 \epsilon_0 \frac{dE}{dt} + \mu_0 \frac{dP}{dt}$$

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$$\nabla^2 E - \mu_0 \epsilon_0 \frac{d^2 E}{dt^2} = \mu_0 \frac{dJ_{free}}{dt} + \mu_0 \frac{d^2 P}{dt^2} - \frac{1}{\epsilon_0} \nabla(\nabla \cdot P)$$

usual wave eqn

electric currents, important when free charges (plasma, metals)

dipole oscillations depends on when no  $J_{free}$

ripples when eq  $\vec{P}$  not in same direction as  $E$

end of day 4