

Lecture 34: Wed, 26 Mar 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. The array theorem and the book's treatment of diffraction gratings both apply to the _____ regime:

- a. Fresnel-Kirkoff
- b. Fresnel
- c. Fraunhofer

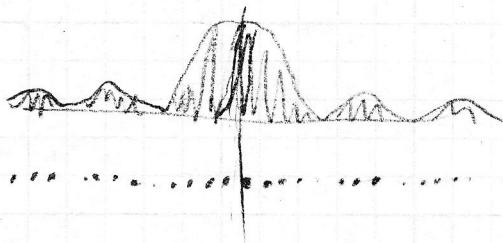
Q2. The array theorem applies to

- a. circular holes only
- b. slits only
- c. any identical apertures

Q3. T/F: The resolving power of a spectrometer depends on the *number* of slits illuminated on the diffraction grating.

Dicussion

Double Slit $I = I_0 \sin^2 \frac{kx a}{2d} \cos^2 \frac{kx h}{2d}$



Maxima given by cos² function

$$\frac{kx h}{2d} = m\pi$$

$$x = \frac{2d}{(2m\pi/\lambda)h} m\pi = \frac{md\lambda}{h}$$

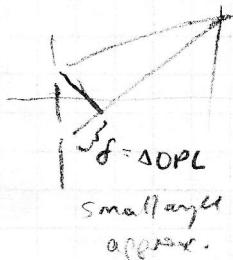
maxima

$$x = \frac{m d \lambda}{h}$$

same as
formula from
106 book

similarly

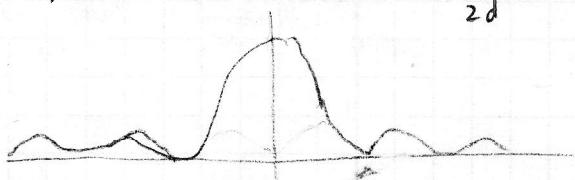
$$\text{minima } x = \left(m + \frac{1}{2}\right) \frac{d\lambda}{h}$$



Note: overall intensity modulated
by sinc function

Single Slit

$$I = I_0 \sin^2 \frac{kx a}{2d}$$



minima & same as sinc function

(except $x=0$ = maxima)

$$\text{minima: } \frac{kx a}{2d} = m\pi$$

$$x = \frac{2d}{(\lambda/\pi)a} m\pi$$

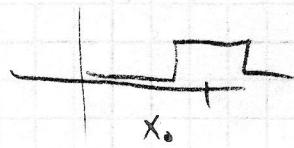
$$\text{minima } x = \frac{m d \lambda}{a}$$

$m \neq 0$

same as 106 book!
(handwriting agreed there)

Note: overall intensity modulated
by sinc function

FT of shifted function

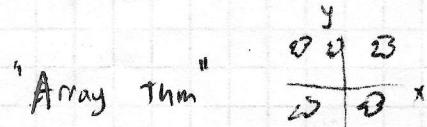


$$\begin{aligned} \text{FT}(\text{rect}(x)) &= \text{FT}(\text{rect}(x)) \otimes \text{FT}\left(\frac{1}{2} \delta(x - x_0)\right) \\ &= \text{sinc} \frac{kx_0 a}{2} \cdot \underbrace{\int_{-\infty}^{\infty} \delta(x - x_0) e^{ikx} dx}_{e^{ikx_0}} \\ &= \text{sinc} \frac{kx_0 a}{2} e^{ikx_0} \end{aligned}$$

(shifts property)

2D: $\boxed{\text{FT (shifted aperture)} = e^{ik_x x_i} e^{ik_y y_0} \cdot \text{FT (regular aperture)}}$

"Array Thm"



$$\text{FT (array)} = \text{FT (aperture)} \cdot \sum_{\text{all apertures}} e^{i(k_x x_{ij} + k_y y_{ij})}$$

Again, not worry about factors of $Y(2\pi)$!

(x_{ij}, y_{ij}) : center of j^{th} aperture

Don't even have to be in an array!

Random vs Non-random array

\sum phase factors
counts

$$I = I(\text{one aperture})$$

\sum phase factors
gives you fine detail

$$I = I(\text{array}) \text{ modulated by } I(\text{one aperture})$$

like \uparrow cosines $\star \text{sinc}^2$

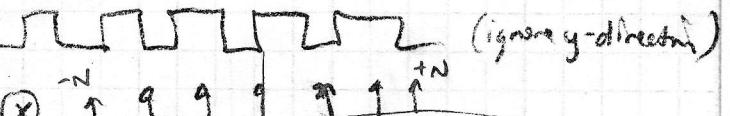
for double slit example

giving 11.7

Diffractn Gratings operative function

$$= \sum_{k=-N}^{+N} \delta(x - k\Delta x)$$

$\max \rightarrow 2\pi$



(ignoring off-crests)

Fourier Transform

$$\frac{1}{2\pi} \sin \frac{kx}{2}$$

$$+ \frac{4}{\sqrt{2\pi}} \sin \left(\frac{kx \Delta x}{2} \right)$$

"comb function" remember?

$2N+1$ total delta functions

$$FT = \frac{1}{\sqrt{2\pi}} \frac{\sin((2N+1)\frac{w\Delta x}{2})}{\sin \frac{w\Delta x}{2}} \quad \begin{matrix} \text{from bandt} \\ w \rightarrow 2\pi \\ \Delta x \end{matrix}$$

let N now = total number

$$= \frac{1}{\sqrt{2\pi}} \frac{\sin N \frac{kx \Delta x}{d}}{\sin \frac{kx \Delta x}{d}}$$

$$= \frac{1}{2\pi} \Delta x \sin \frac{kx \Delta x}{2d} \frac{\sin \frac{N kx \Delta x}{d}}{\sin \frac{kx \Delta x}{d}}$$

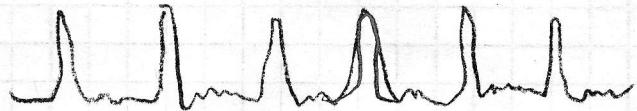
$$I = I_0 \sin^2 \frac{kx \Delta x}{2d} \frac{\sin^2 \frac{N kx \Delta x}{d}}{N^2 \sin^2 \frac{2 kx \Delta x}{d}}$$

↑
added N^2 in denominator (and removed Δx from numer.)
so $I(x=0) = I_0$

recall $\frac{\sin Nx}{Nx} \rightarrow N$ as $x \rightarrow 0$

also from delta function bandt

Recall $\frac{\sin Nx}{Nx}$ plotted in delta function bandt for $N=10$



Now \uparrow squared

multiplied by \sin^2

this peak happens to fall on minima of \sin^2 (because delta function spacing = $2\Delta x$ exactly)

↑ position of this

"first order peak" (and other peaks)

depends on wavelength
(via k)

Multiplex
Animator

section 11.8

Spectrometer

use (first order) diffraction spot to give you wavelength separation

let grating spacing = h now

instead of $2\Delta x$

- 1) position of peaks set by comb function
- 2) amplitude of peaks set by sinc function

→ peaks occur when denominator goes to zero $\sin x = 0 \rightarrow x = m\pi$

$$\frac{k_x \times (h/2)}{d} = m\pi$$

$$\frac{2\pi}{\lambda} \times \frac{(h/2)}{d} = m\pi$$

$$\lambda = \frac{x h}{m d}$$

x = position on screen

d = distance from grating to screen

h = grating spacing (meters/line/mm)

Again from delta function handout

$$\text{distance (in } k\text{-space) to first zero} = \frac{2\pi}{N h}$$

$$\Delta k_x = \frac{2\pi}{N h}$$

$$\frac{x}{d} \Delta k = \frac{2\pi}{N h}$$

$$\frac{x}{d} \frac{2\pi N d}{\lambda^2} = \frac{2\pi}{N h}$$

$$\Delta \lambda = \frac{d}{x} \frac{\Delta \lambda}{N h}$$

$$\frac{x h}{d} = m \lambda$$

$$= \frac{1}{m N} \frac{\lambda^2}{N}$$

$$\boxed{\Delta \lambda = \frac{\lambda}{m N}}$$

close to $\Delta \lambda_{FWHM}$

$$\Delta k_x = \frac{k_x}{d}$$

$$k = \frac{2\pi}{\lambda}$$

$$\Delta k = \frac{2\pi}{\lambda^2} \Delta \lambda$$



$$R P = \frac{\lambda}{\Delta \lambda}$$

$$\boxed{R P = m N}$$

increase RP by increasing N (# lines illuminating light)
and/or m (but this decreases intensity)

Lecture 35: Fri, 28 Mar 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. Bessel functions are used in diffraction cases of _____ symmetry

- a. square
- b. rectangular
- c. cylindrical
- d. spherical

Q2. T/F: The function $J_0(x)$ crosses zero at $\pi, 2\pi, 3\pi$, etc.

Q3. T/F: Dr. Colton's handouts have some awesome pictures in them.

Cylindrical coords, only f(r)

$$F(k_r, k_\theta) = \iint f(r) e^{i(k_x x' + k_y y')} dx' dy'$$

Possibly
factor of $\frac{1}{2\pi}$ $x = r \cos\theta$

$$k_x = k_r \cos k_\theta$$

$$y = r \sin\theta$$

$$r = \sqrt{x^2 + y^2}$$

$$k_r = \sqrt{k_x^2 + k_y^2}$$

and $dx' dy' = r dr d\theta$ (book uses r instead of dr) (Cyl. uses s)

$$F(k_r, k_\theta) = \iint f(r) e^{i[(k_r \cos k_\theta)(r \cos\theta) + (k_r \sin k_\theta) r \sin\theta]} r dr d\theta$$

$$= \iint f(r) e^{ik_r r [\cos k_\theta \cos\theta + \sin k_\theta \sin\theta]} r dr d\theta$$

$$= \int_0^\infty f(r) r \left[\int_0^{2\pi} e^{ik_r r [\cos(\theta - k_\theta)]} d\theta \right] dr$$

The integral in the brackets belongs to a class of functions called the Bessel functions, defined by

$$J_\alpha(x) = \frac{1}{\pi} \int_0^\pi \cos(\alpha\theta - x \sin\theta) d\theta$$

Specifically, we have an $\alpha=0$ Bessel function, since

$$J_0(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin\theta) d\theta$$

Mark

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x \sin \theta) d\theta \quad \text{since } \cos(x \sin \theta) = \text{even}$$

proof: $\cos(x \sin(-\theta)) = \cos(-x \sin \theta) = \cos(x \sin \theta)$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} \cos(x \cos(\theta - \frac{\pi}{2})) d\theta \quad \text{since } \sin \theta = \cos(\theta - \frac{\pi}{2})$$

$$= \frac{1}{2\pi} \int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} \cos(x \cos \theta) d\theta$$

$\theta' = \theta - \frac{\pi}{2}$ subst. ; then rename back to θ

~~$\frac{1}{2\pi} \int_0^{2\pi} \cos(x \cos \theta) d\theta$ because integrating over full period both ways~~

$$= \frac{1}{2\pi} \int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}} [\cos(x \cos \theta) + i \sin(x \cos \theta)] d\theta$$

since the sine part integrates to 0

proof: $-\frac{3\pi}{2}$ to $-\frac{\pi}{2}$ part + cancels w/ $\frac{-\pi}{2}$ to $\frac{\pi}{2}$
since H's "odd" with respect to $\frac{\pi}{2}$

i.e. $\sin(x \cos(-\frac{\pi}{2} - \theta)) = -\sin(x \cos(\frac{\pi}{2} + \theta))$

obviously true (?) since

$$\cos(-\frac{\pi}{2} - \theta) = \cos(-\frac{\pi}{2} + \theta)$$

just look at cosine graph

$$= \frac{1}{2\pi} \int_{-3\pi/2}^{\pi/2} e^{ix \cos \theta} d\theta$$

$$= \frac{1}{2\pi} \int_{-\pi/2}^{\pi/2} e^{ix \cos \theta} d\theta \quad \text{because still integrating over a full period}$$

do $\theta' = \theta + k_0$, then rename back to θ

$$= \frac{1}{2\pi} \int_0^{\pi} e^{ix \cos(\theta - k_0)} d\theta$$

11

so this stuff = $2J_0(x)$

i.e. $F(k_r, k_\theta) = 2\pi \int_0^\infty f(r) J_0(k_r r) r dr$

i) FT of radial functn, f .

$$\text{or } J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\sin\theta} d\theta$$

because $\int_0^{2\pi} \sin[x\sin\theta] d\theta = 0$
and cosine is even

This also can be written as

$$J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos\theta} d\theta$$

$$\text{or even } J_0(x) = \frac{1}{2\pi} \int_0^{2\pi} e^{ix\cos(\theta + \phi)} d\theta$$

for any ϕ since we're integrating over a complete period.

Thus we have

$$F(kr, k_\theta) = 2\pi \int_0^\infty f(r) J_0(rkr) r dr$$

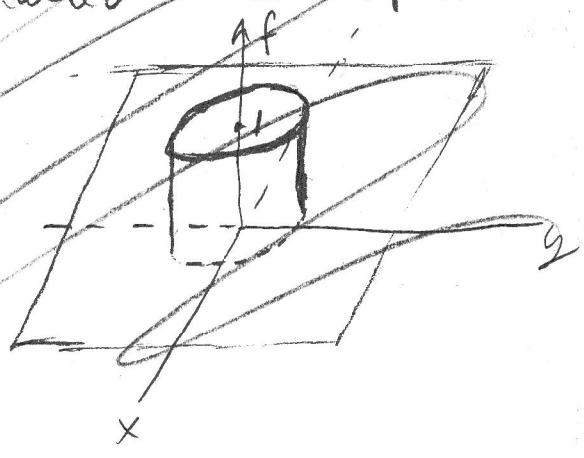
factor of
 2π ?

Fourier transform for a radial function f .

We can now apply this expression to a simple function, sometimes called the "top hat" function:

$$f(x, y) = \begin{cases} 1 & \sqrt{x^2 + y^2} \leq 1 \\ 0 & \text{all other } x, y \end{cases}$$

$$\text{or } f(r) = \begin{cases} 1 & |r| \leq 1 \\ 0 & \text{all other } r \end{cases}$$

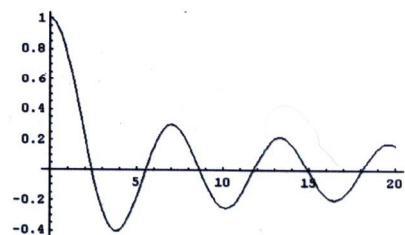


WTF

Bessel Function Plots

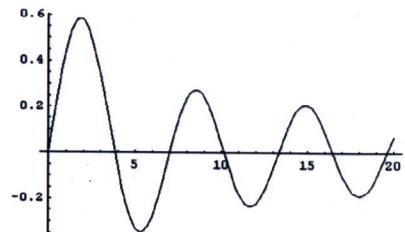
by John S. Colton

In[25]:= Plot[BesselJ[0, x], {x, 0, 20}]



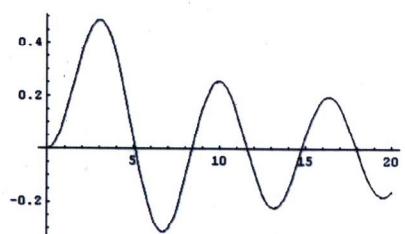
Out[25]= - Graphics -

In[26]:= Plot[BesselJ[1, x], {x, 0, 20}]



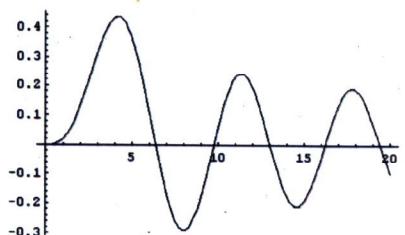
Out[26]= - Graphics -

In[27]:= Plot[BesselJ[2, x], {x, 0, 20}]



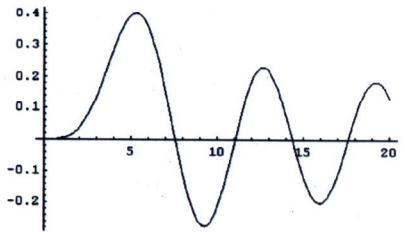
Out[27]= - Graphics -

In[28]:= Plot[BesselJ[3, x], {x, 0, 20}]



Out[28]= - Graphics -

In[29]:= Plot[BesselJ[4, x], {x, 0, 20}]



Out[29]= - Graphics -

BesselJ[α , x] is built into *Mathematica*, just like Sin[x]

$J_0(x)$

crosses zero at 2.405, 5.520, 8.654, ...

← The only one that is not zero at the origin

$J_1(x)$

crosses zero at 3.832, 7.016, 10.173, ...

$$J_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (k+n)!} \frac{x^{k+n}}{2^k k!}$$

$J_2(x)$

crosses zero at 5.136, 8.417, 11.620, ...

$J_3(x)$

crosses zero at 6.380, 9.761, 13.015, ...

$J_4(x)$

crosses zero at 7.588, 11.065, 14.373, ...

Bessel Functions vs. Sines/Cosines

oscillatory

Sines/Cosines

- Two functions: $\sin x$ and $\cos x$
(Sometimes $\sin x$ or $\cos x$ is not used, due to the symmetry of the problem.)
- You determine the value of $\sin x$ or $\cos x$ for arbitrary x by using a calculator or computer program

$$3. \quad \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!}$$

Consider just $\sin x$:

- The zeroes of $\sin x$ are at $x = \pi, 2\pi, 3\pi, \dots$
 $x = "m\pi"$ is the m^{th} zero
- At $x = 1$, $\sin(m\pi x) = 0$ for all m
- The differential equation satisfied by $f = \sin(m\pi x)$ is
$$f'' + (m\pi)^2 f = 0$$
- $\sin(n\pi x)$ is orthogonal to $\sin(m\pi x)$ on the interval $(0,1)$:

$$\int_0^1 \sin(n\pi x) \sin(m\pi x) dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{1}{2}, & \text{if } n = m \end{cases}$$

Additionally, the Bessel functions are related to sines/cosines through this integral formula:

$$J_\alpha(x) = \frac{1}{\pi} \int_0^\pi \cos(\alpha\theta - x \sin \theta) d\theta$$

Boas: "In fact, if you had first learned about $\sin(nx)$ and $\cos(nx)$ as power series solutions of $y'' = -n^2 y$, instead of in elementary trigonometry, you would not feel that Bessel functions were appreciably more difficult or strange than trigonometric functions. Like sines and cosines, Bessel functions are solutions of a differential equation; they are tabulated and their graphs can be drawn; they can be represented as a series; and a large number of formulas about them are known."

by John S. Colton

oscillatory

Bessel functions

Two functions for each α : $J_\alpha(x)$ and $N_\alpha(x)$
(Typically N_α is not used because it's infinite at the origin.)

You determine the value of $J_\alpha(x)$ for arbitrary x by using a calculator or computer program

$$J_\alpha(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+\alpha}}{k!(k+\alpha)! 2^{2k+\alpha}}$$

Consider just $J_\alpha(x)$ for one α , say $\alpha = 0$:

(similar things hold true for all α 's)

The zeroes of $J_0(x)$ are at
 $x \approx 2.405, 5.520, 8.654, \dots$
 $x = "u_{0m}"$ is the m^{th} zero

At $x = 1$, $J_0(u_{0m}x) = 0$ for all m

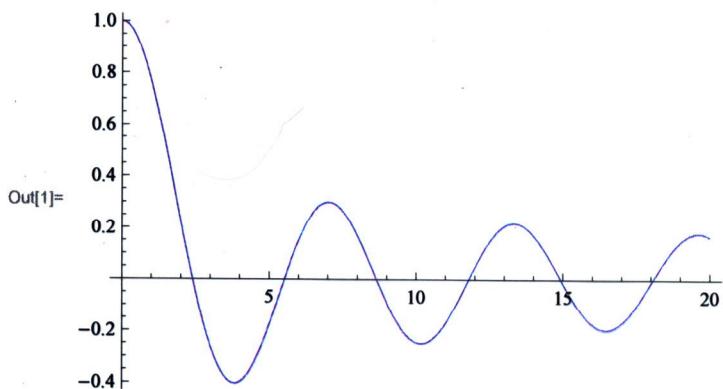
The differential equation satisfied by $f = J_0(u_{0m}x)$ is
$$x^2 f'' + x f' + (u_{0m}^2 x^2 - 0^2) f = 0$$

 $0^2 \rightarrow \alpha^2$ for other α 's

$J_0(u_{0n}x)$ is orthogonal to $J_0(u_{0m}x)$ on the interval $(0,1)$, with respect to a "weighting" of x :

$$\int_0^1 x J_0(u_{0n}x) J_0(u_{0m}x) dx = \begin{cases} 0, & \text{if } n \neq m \\ \frac{1}{2} (J_1(u_{0m}))^2, & \text{if } n = m \end{cases}$$

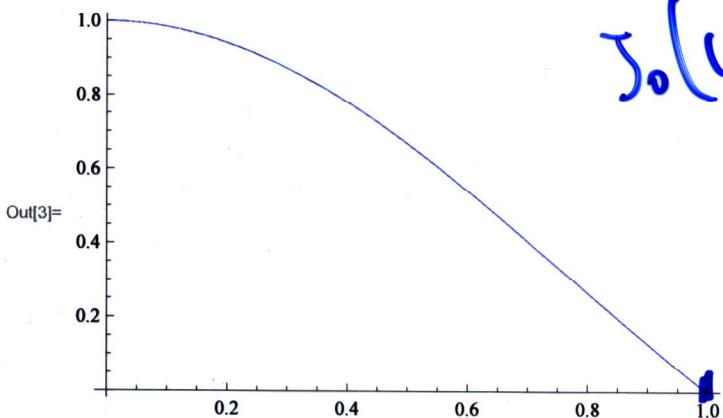
```
In[1]:= Plot[BesselJ[0, s], {s, 0, 20}]
```



```
In[2]:= f[x_] = BesselJ[0, 2.405 x]
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Out[2]= BesselJ[0, 2.405 x]
```

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In[3]:= Plot[f[x], {x, 0, 1}]
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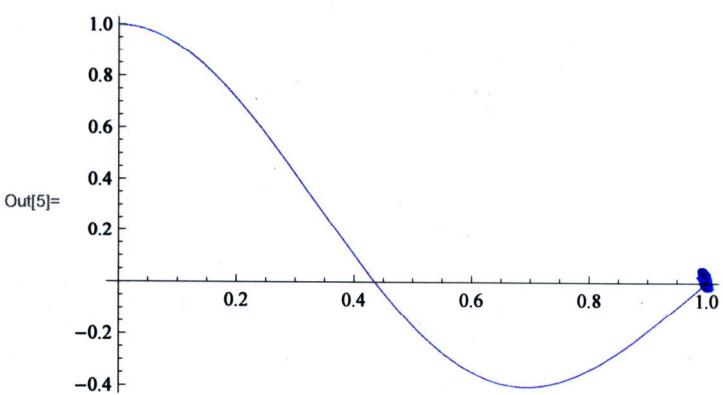


$$2.405x$$
$$J_0(u_{01}x)$$

```
In[4]:= g[x_] = BesselJ[0, 5.520 x]
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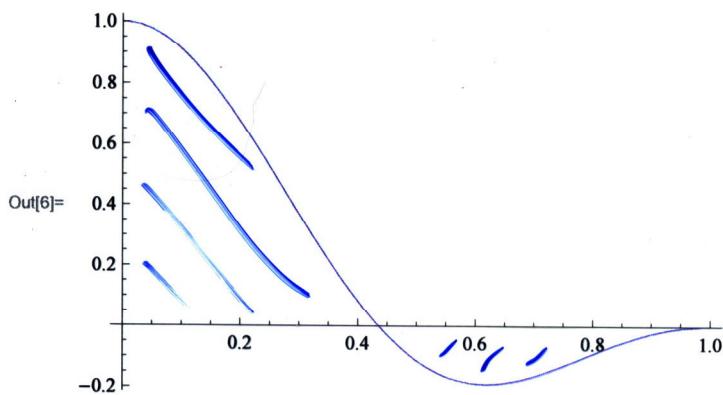
```
Out[4]= BesselJ[0, 5.52 x]
```

```
In[5]:= Plot[g[x], {x, 0, 1}]
```

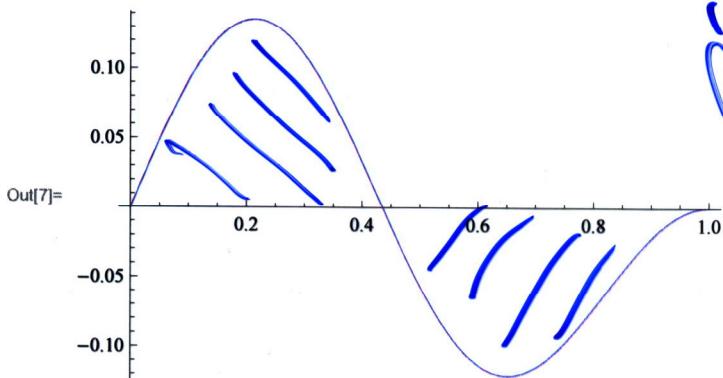


$$5.52x$$
$$J_0(u_{02}x)$$

In[6]:= Plot[f[x] g[x], {x, 0, 1}]



In[7]:= Plot[x f[x] g[x], {x, 0, 1}]

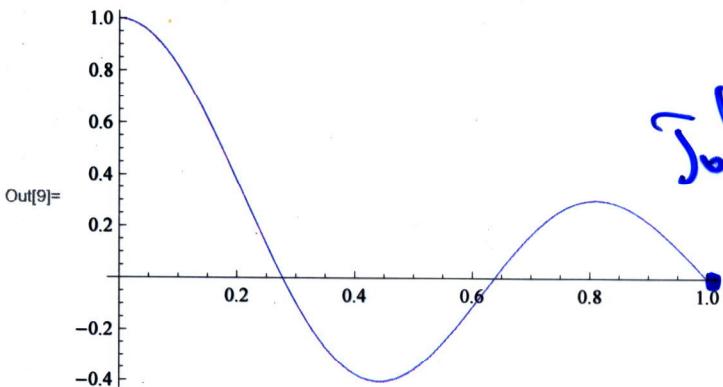


$$\int_0^1 x J_0(u_0 x) J_0(u_{02} x) dx = 0$$

In[8]:= h[x_] = BesselJ[0, 8.654 x]

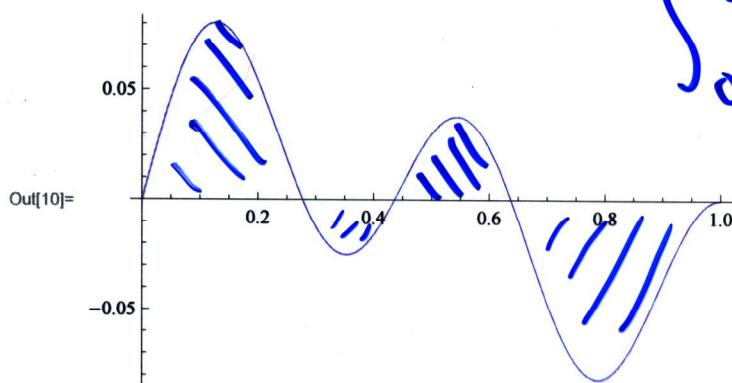
Out[8]= BesselJ[0, 8.654 x]

In[9]:= Plot[h[x], {x, 0, 1}]



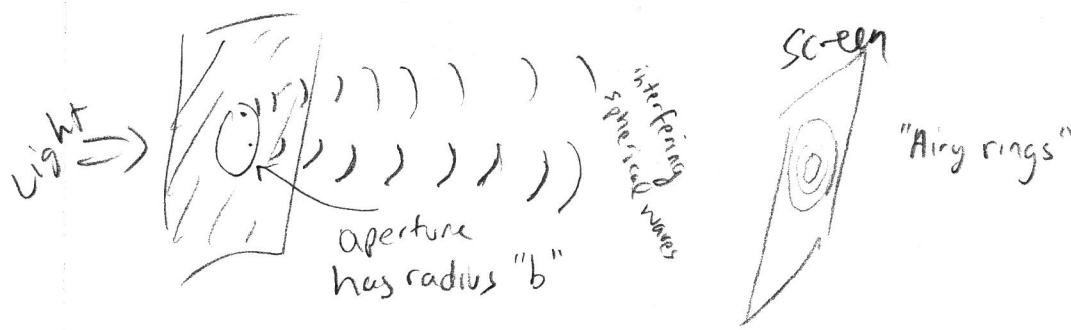
$$J_0(8.654 x)$$

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In[10]:= Plot[x g[x] h[x], {x, 0, 1}]
```



$$\int_0^1 J_0(u_2 x) J_0(v_2 x) + dt = 0$$

Diffraction from a circular aperture



The aperture function is ~~what is~~ exactly the same "top hat function" ~~we just looked at~~.

~~We go through the same process~~

$$g_{\text{aperture}}(\text{aperture}) = 2\pi \int_0^{\infty} (\text{aperture function}) r J_0(r k_r) dr$$

F.T. for polar coordinates

$$= 2\pi \int_0^b r J_0(r k_r) dr$$

$$\text{let } r' = r k_r$$

$$= 2\pi \int_0^{b k_r} \left(\frac{r'}{k_r} \right) J_0(r') \left(\frac{dr'}{k_r} \right)$$

$$= \frac{2\pi}{k_r^2} \underbrace{\int_0^{k_r b} r' J_0(r') dr'}_{(k_r b) J_1(k_r b)}$$

$$(k_r b) J_1(k_r b)$$

$$= \frac{2\pi b}{k_r} J_1(k_r b)$$

$$g_{\text{aperture}}(\text{top hat of radius } b) = 2\pi b^2 \frac{J_1(k_r b)}{k_r b}$$

$J_1(x) = "jinc"$, sometimes

WNA

PWN

or $\frac{2J_1(x)}{x} = "jinc"$
so that $jinc(0) = 1$
(like sinc)

Recall that the diffraction pattern varies as the F.T., squared, with $k_r = \frac{kR}{d}$
 $k_r = \sqrt{k_x^2 + k_y^2} = \frac{2\pi}{\lambda} \sqrt{x^2 + y^2}$
 (R = radial distance on the screen, d = distance the screen is away from the aperture,
 $k = \frac{2\pi}{\lambda}$)

So the diffraction pattern seen on the screen is: (shape, at least)

$$I = I_0 \left[\frac{2 J_1 \left(\frac{kRb}{d} \right)}{\left(\frac{kRb}{d} \right)} \right]^2$$

factor of 2
 so that $I_0 = I(R=0)$
 since $J_1(x) = \frac{1}{2} e^{jx} - j^2 e^{-jx}$

This is the "Airy ring pattern", ~~which~~
~~you likely studied at optics.~~

$I-I$'s plotted on the next page ...

also, E had λd in denominator
 more 2nd order

$$\text{Books answer } I = I_0 \left(\frac{1}{\lambda d} \right)^2 \left(\frac{2\pi b^2}{\lambda d} \right)^2 \left(\frac{J_1 \left(\frac{kRb}{\lambda d} \right)}{\frac{kRb}{\lambda d}} \right)^2$$

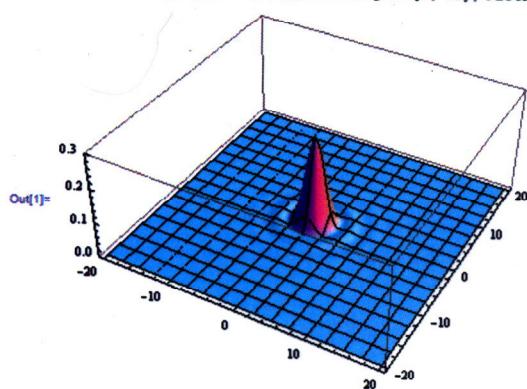
WRB

$$\frac{2 J_1(x)}{x} = \sin(x)$$

Fourier Transform of “Top Hat” Function, squared

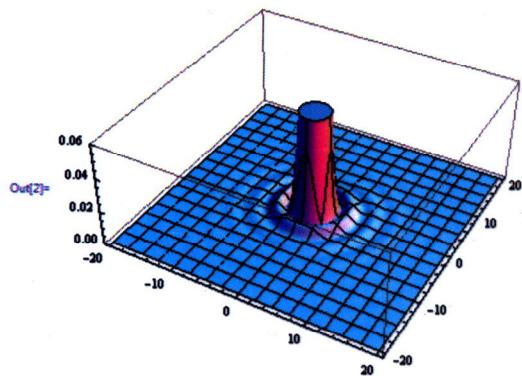
by John S. Colton

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In[1]:= Plot3D[(BesselJ[1, (x^2 + y^2)^(1/2)] / (x^2 + y^2)^(1/2))^2,
{x, -20, 20}, {y, -20, 20}, PlotRange -> {0, .3}, PlotPoints -> 100]
```

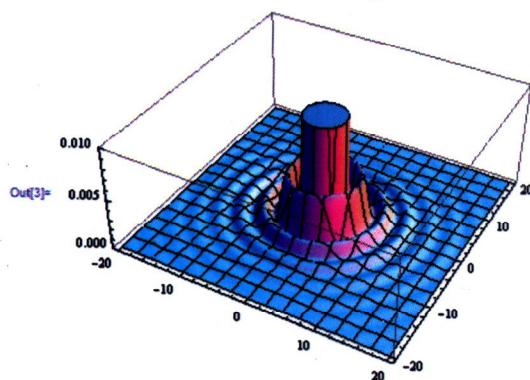


$$I = I_0 \left(\frac{J_1(\rho)}{\rho} \right)^2; \quad \rho = \sqrt{x^2 + y^2}$$

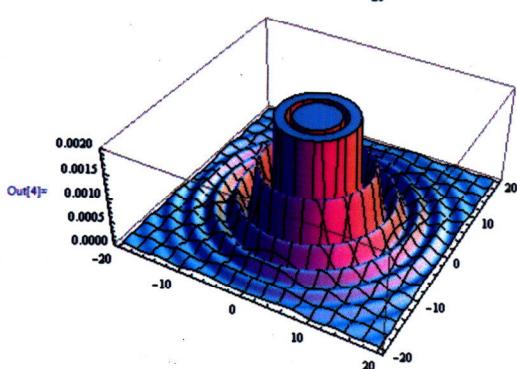
This is called the “Airy pattern”; the central maximum is called the “Airy disk”.



zoomed in



zoomed in further



zoomed in still further

This is the far-field diffraction pattern you get from a circular opening!

(If it results from a circular blockage rather than a circular opening, the central peak would be called the “Poisson spot”.)

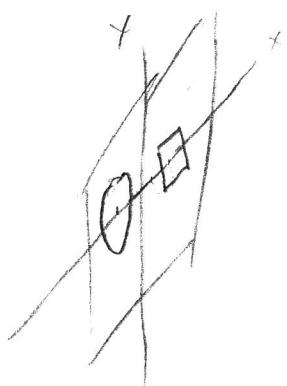
Aperture shifted from origin \rightarrow very much like array theorem

$$\text{F.T. of shifted aperture} = e^{ik_x x_0} e^{ik_y y_0} \times \text{F.T. of regular aperture}$$

} already discussed

So shifting the origin just introduces phase factors. Not terribly important for this example, but can be if you have different phase factors from multiple apertures.

Example Problem Diffracton pattern of circle



Square combo, given ...

circle : radius a

square : side a

square + circle each centred a distance
"2a" from the origin.

We can immediately use the ^{previous} results and write down the F.T.

$$\text{F.T.} = (\text{F.T.})_{\text{shifted square}} + (\text{F.T.})_{\text{shifted circle}}$$

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