Lecture 5: Wed, 16 Jan 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. True/False – Students need to know the material in section 0.2 on complex numbers by heart.

Q2. Complete "Euler's formula". $e^{i\theta} = \ldots$ $\cos\theta + i\sin\theta$

Q3. In "Colton's Complex Number Summary", four ways of writing a given complex number are mentioned. Name three of them.

$x + iy$ rect

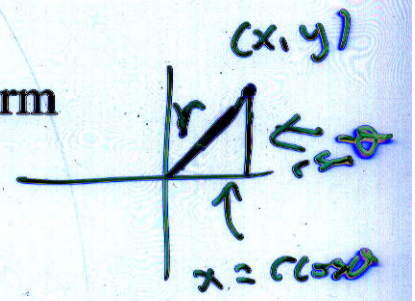
(r, θ) polar

$re^{i\theta}$ exponential

$r\angle\theta$ phasor

Complex numbers: a summary by Dr. Colton

(x, y)



- $x + iy$ can be written in rectangular or polar form
  - use rectangular for adding/subtracting
  - use polar form for multiplying/dividing
- Euler's equation: $e^{i\theta} = \cos\theta + i\sin\theta$
- polar form $(A, \theta) \Leftrightarrow$ complex exponential $Ae^{i\theta}$
- Example: $3 + 4i$:

  $= (3, 4)$ in rectangular form,

  $= (5, 53.13°)$ in polar form, and $\quad 5e^{i(53.13°)}$

  $= 5e^{0.9273i}$ complex exp. form ($53.13° = 0.9273$ rad)
- multiply: $A_1e^{i\theta_1} \times A_2e^{i\theta_2} = A_1A_2e^{i(\theta_1+\theta_2)} = (A_1A_2,\ \theta_1+\theta_2)$
- divide: $A_1e^{i\theta_1} \div A_2e^{i\theta_2} = (A_1/A_2)e^{i(\theta_1-\theta_2)} = (A_1/A_2,\ \theta_1-\theta_2)$
- phasor form: $A\angle\theta$ (just like polar form, for now)
  - "$\angle$" = "at an angle of"
  - Example: $(3 + 4i) \times (5 + 12i)$

    $= 5\angle 53.13° \times 13\angle 67.38°$

    $= 65\angle 120.51°$

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In[1]:=Plot[5 Cos[3 x] + 4 Sin[3 x],{x,0,5}]
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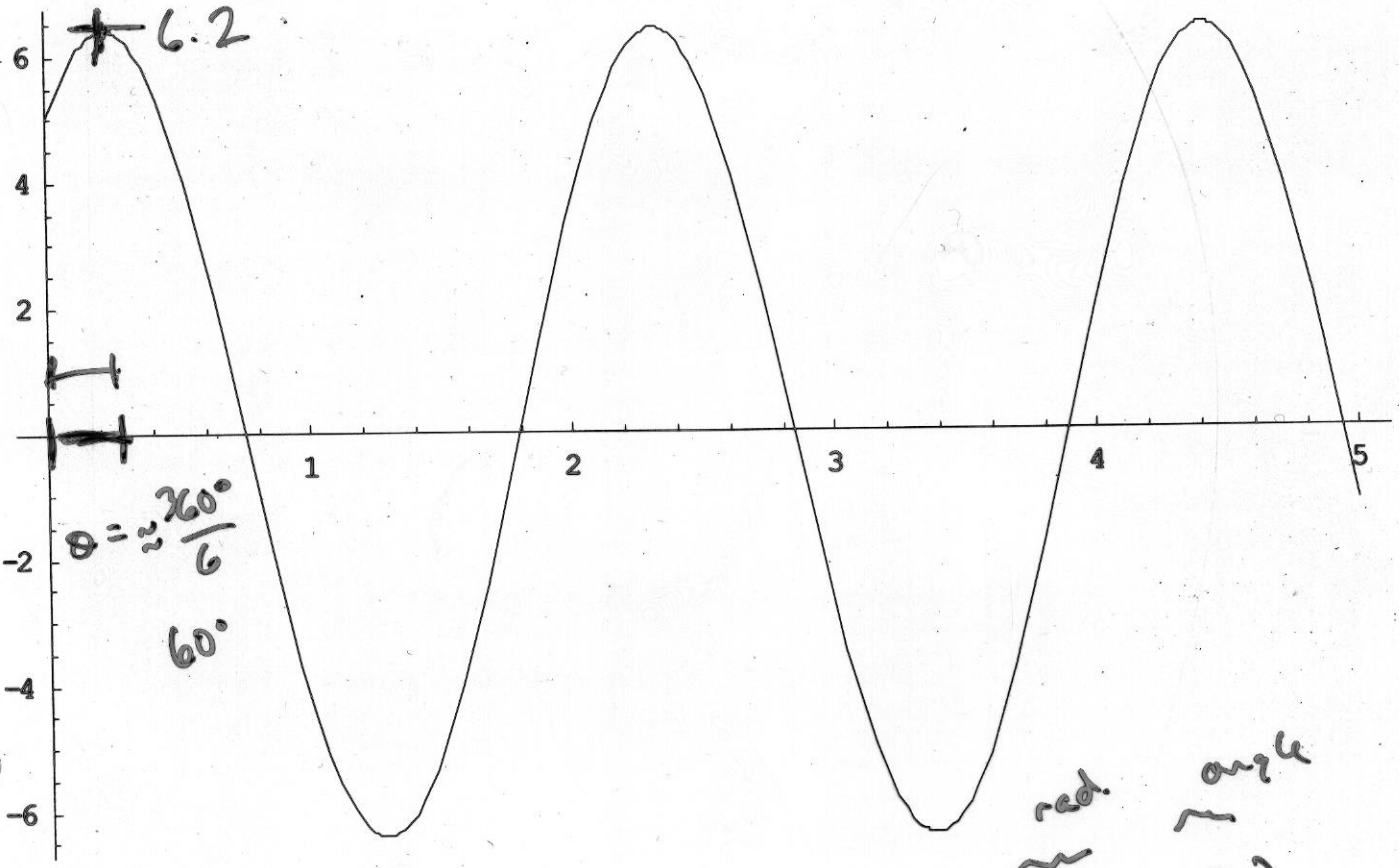


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Out[1]= -Graphics-
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$$6.2 \cos(\overbrace{3x}^{\text{rad.}} - \overbrace{60°}^{\text{angle}})$$

$$(3(x - .4))$$

Mathematica survey

day 5

Complex Numbers

→ handout!

$$\boxed{\text{Euler} \quad e^{i\theta} = \cos\theta + i\sin\theta}$$

$$Ae^{i\theta} = A\cos\theta + iA\sin\theta$$

Relating the other way

so $$\boxed{A\cos\theta = \text{Real}\left[Ae^{i\theta}\right]}$$

Sinusoidal function varying in time

$$5\cos(\omega t + 30^\circ) = \text{Re}\left[5\, e^{i(\omega t + 30^\circ)}\right]$$

$$= 5e^{i30^\circ} e^{i\omega t}$$

$$= 5\angle 30^\circ$$

"phasor notation"

- real part assumed, (handout [illegible])

~~when all terms in an equation have same eg. time dependence, we don't have to worry about them (ie they all cancel out)~~

$$= 5\angle 30^\circ$$

- $e^{i\omega t}$ assumed, $\boxed{\text{can do if all terms in eqn will have it}}$

~~This does: $e^{i\omega t}$ assumed (rather than $\cos\omega t$)~~

~~Since traveling waves will be like $\cos(kx - \omega t)$~~

later

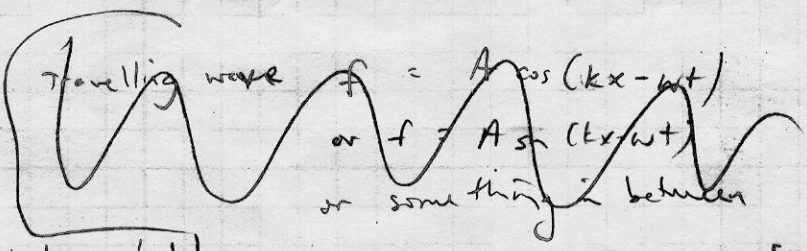
Note

Application

Travelling wave $f = A\cos(kx - \omega t)$

or $f = A\sin(kx - \omega t)$

or something in between



Back to $\cos(\omega t)$

Fact: any linear comb of sine + cosine (same freq.) is also "sinusoidal", but w/ different phase (as long as same k and ω)

How do you figure out what phase is?

→ complex numbers! (make life much easier)

problem: $5\cos 3t + 4\sin 3t = ___\cos(3t + ___)$ Overhead

graph: amplitude $\simeq 6.2$

phase $\simeq -30°$

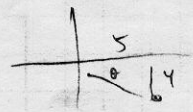
$5\angle 0° + 4\angle -90°$

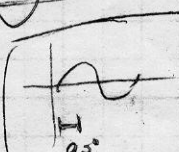
always understood $\mathrm{Re}[\quad]$

adding → convert to rectangular form

$5 + 0i + 0 - 4i$

$5 - 4i$





90°

why sine wave has $\phi = -90°$

now back to polar form

$= 6.403\angle -38.66°$

$6.403\, e^{-i38.66°}$

- understood radians converted
- understood e^{i3t}

Convert back to cosine notation

$6.403\cos(3t - 38.66°)$

Challenge: work out this answer some other way! Bonus pts for first person

Optics → an understood $e^{i(kx - \omega t)}$

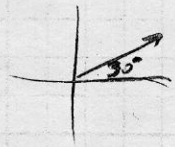
"Complex amplitude" → this is basically the phasor

ex: $\vec{E} = 8\cos(kz - \omega t + 30^\circ)\,\hat{x}$

$\vec{E}_0$ = amplitude = 8 or $8\hat{x}$

complex amplitude = $8\angle 30^\circ\ \hat{x}$

$= 8\cos 30^\circ + 8i\sin 30^\circ$



Notation: regular amplitude → regular vector symbol $\vec{E}_0$

complex amplitude → new vector symbol $\tilde{E}_0$

(Griffiths notation; P+W don't use ~)

Driven/Damped Harmonic Oscillator – Lorentz model

Slight

$F = qE$

$$\Sigma F = m\ddot{x}$$

N-eqm.q

$$F_{driving} + F_{spring} + F_{damping} = m\ddot{x}$$

$$qE_0 \cos(-\omega t) - kx - \gamma m\dot{x} = m\ddot{x}$$

$$\ddot{x} + \gamma\dot{x} + \frac{k}{m}x = \frac{qE_0}{m}\cos(-\omega t)$$

$\rightarrow$ Let $\omega_0 = \sqrt{\frac{k}{m}}$

$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0{}^2 x = \frac{qE_0}{m}\cos(-\omega t)}$$

equation of motion

Guess $x = x_0 \cos(-\omega t + \phi)$ as solution

Complex numbers: $\tilde{x} = \tilde{x}_0 e^{i(-\omega t)}$ (ϕ lumped in with $\tilde{x}_0$)

Derivatives: $\dot{\tilde{x}} = (-i\omega)\tilde{x}_0 e^{i(-\omega t)}$

$$\ddot{\tilde{x}} = (-i\omega)^2 \tilde{x}_0 e^{i(-\omega t)}$$

Plug into boxed equation:

$$(-i\omega)^2 \tilde{x}_0 \cancel{e^{i(-\omega t)}} + \gamma(-i\omega)\tilde{x}_0 \cancel{e^{i(-\omega t)}} + \omega_0{}^2 \tilde{x}_0 \cancel{e^{i(-\omega t)}} = \frac{qE_0}{m}\cancel{e^{i(-\omega t)}}$$

Divide by $e^{i(-\omega t)}$, factor out $\tilde{x}_0$: $\tilde{x}_0\left(-\omega^2 + \gamma(-i\omega) + \omega_0{}^2\right) = \frac{qE_0}{m}$

$$\boxed{\tilde{x}_0 = \frac{qE_0}{m}\frac{1}{\omega_0{}^2 - \omega^2 - i\omega\gamma}}$$ The answer!!

Or more specifically, $\tilde{x} = \frac{qE_0}{m}\frac{1}{\omega_0{}^2 - \omega^2 - i\omega\gamma}e^{-i(wt)}$

$e^{i(wt+\phi)}$

$$\frac{1}{1+\sqrt{2}} \quad \frac{1-\sqrt{2}}{1-\sqrt{2}}$$

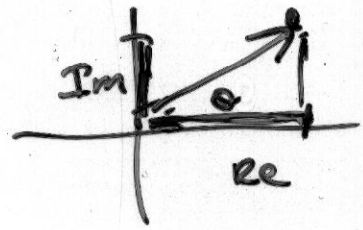
Convert to polar form to get phase & amplitude

(need real + imaginary)

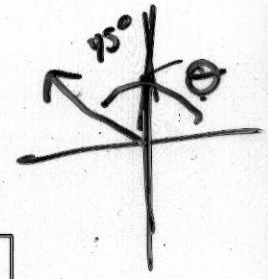
$$\frac{1}{\omega_0^{\,2}-\omega^2-i\omega\gamma}\times\frac{\omega_0^{\,2}-\omega^2+i\omega\gamma}{\omega_0^{\,2}-\omega^2+i\omega\gamma} = \frac{\omega_0^{\,2}-\omega^2+i\omega\gamma}{\left(\omega_0^{\,2}-\omega^2\right)^2+(\omega\gamma)^2}$$

Real

Real part of $\tilde{x}_0 = \dfrac{qE_0}{m}\dfrac{\omega_0^{\,2}-\omega^2}{\left(\omega_0^{\,2}-\omega^2\right)^2+(\omega\gamma)^2}$



Imaginary part $= \dfrac{qE_0}{m}\dfrac{\omega\gamma}{\left(\omega_0^{\,2}-\omega^2\right)^2+(\omega\gamma)^2}$



Need to convert to polar form: $\sqrt{\mathrm{Re}^2+\mathrm{Im}^2}\angle\theta$

A little algebra: $\boxed{\sqrt{\mathrm{Re}^2+\mathrm{Im}^2} = \dfrac{qE_0}{m}\dfrac{1}{\sqrt{\left(\omega_0^{\,2}-\omega^2\right)^2+(\omega\gamma)^2}}}$

$$\boxed{\theta = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^{\,2}-\omega^2}\right)}$$

(or possibly $\underbrace{-\tan^{-1}}_{\text{Not correct}}$**, if in II or III quadrant)**

Mathematica: use "Arg" command to avoid this possible error

Final answer:

$$\boxed{x(t) = \frac{qE_0}{m}\frac{1}{\sqrt{\left(\omega_0^{\,2}-\omega^2\right)^2+(\omega\gamma)^2}}\cos\left(-\omega t+\tan^{-1}\left(\frac{\omega\gamma}{\omega_0^{\,2}-\omega^2}\right)\right)}$$

(or $\underbrace{-\tan^{-1}}_{\text{Not correct}}$ **if in II or III quadrant)**

Lecture 6: Fri, 18 Jan 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. True/false: the magnetic field can be ignored in most optics problems.

Q2. The electric susceptibility χ bears which relation to the electric field and polarization:

a. $\boldsymbol{P} = \varepsilon_0 \chi \boldsymbol{E}$
b. $\cancel{\boldsymbol{P}} = \varepsilon_0 (1+\chi) \boldsymbol{E}$ $= \vec{D}$ $\quad \epsilon_r = 1 + \chi$
c. $\boldsymbol{P} = \varepsilon_0 \boldsymbol{E}/\chi$
d. $\boldsymbol{P} = \varepsilon_0 \boldsymbol{E}/(1+\chi)$

Q3. In a material with electrons that have different transition frequencies, the strength of each transition's contribution to the optical properties is called the _____ strength.

a. oscillator
b. mixed
c. frequency
d. transition

Back to Maxwell's Eqns!

motion in the x-direction $\cos(kx - \omega t)$

$\omega = \frac{2\pi}{\text{period}}$

$k = \frac{2\pi}{\text{wavelength}}$

motion in the $\vec{r}$ direction $\cos(\vec{k}\cdot\vec{r} - \omega t)$

$(k_x x + k_y y + k_z z - \omega t)$

direction of $\vec{k} = \hat{k}$ unit vector in (k_x, k_y, k_z)

$\hat{k}$ = dir. of propogation

Add phase:

$$\vec{E} = \vec{E}_0 \cos(\vec{k}\cdot\vec{r} - \omega t + \phi)$$

complex: $\tilde{E} = \tilde{E}_0 e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

$$\tilde{E}_0 = \vec{E}_0 e^{i\phi}$$

side note: plane waves vs. eg spherical waves

$\cos(kz - \omega t)\hat{x}$ $\qquad$ $\frac{1}{r}\cos(kr - \omega t)\hat{r}$

but focus on plane waves

become plane-like at large distances

~~plane waves:~~

~~- Stokes, byu.edu / ~~concave~~ - flash, textbook~~

~~- cover of textbook~~

Note: • E + B perpendicular to each other

• " " " to $\vec{k}$

• E + B in phase w/ each other

Why?

day 6

Assume plane waves (going in z-direction for simplicity)

↓

i.e. lining z-axis so it's in direction of travel

$\tilde{E} = \tilde{E}_0 e^{i(kz - \omega t)}$

$\tilde{B} = \tilde{B}_0 e^{i(kz - \omega t)}$

$\tilde{B}_0$ could have diff phase than $\tilde{E}_0$

they (obviously?) satisfy wave eqn, but do they satisfy individual M.E.s?

Make all vectors into tilde

In regions w/ no ρ

$\vec{\nabla}\cdot\vec{E} = 0$

just convert to z-direction solution

$$\nabla\cdot E = \cancel{\frac{\partial E_x}{\partial x}} + \cancel{\frac{\partial E_y}{\partial y}} + \frac{\partial E_z}{\partial z} = 0$$

$$ik\,\tilde{E}_{0z}e^{i(kz-\omega t)} = 0$$

not always 0

must be 0 !

$E_{0z} = 0 \Longrightarrow$ wave has no z-component !

E = transverse

$\nabla\cdot B = 0 \rightarrow \quad B_z = 0 \qquad B$ = transverse

$$\nabla\times E = -\frac{\partial B}{\partial t} \qquad \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = -(-i\omega)\vec{B}$$

$$\hat{x}\left(\cancel{\frac{\partial E_z}{\partial y}} - \frac{\partial E_y}{\partial z}\right) - \hat{y}\left(\cancel{\frac{\partial E_z}{\partial x}} - \frac{\partial E_x}{\partial z}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) = i\omega\left(\hat{x}B_x + \hat{y}B_y + \cancel{\hat{z}B_z}\right)$$

$\frac{\partial}{\partial z} \rightarrow ik$

$\hat{x}: \quad -\frac{\partial E_y}{\partial z} = i\omega B_x \rightarrow -\cancel{i}k E_y = \cancel{i}\omega B_x$

$\hat{y}: \quad +\frac{\partial E_x}{\partial z} = i\omega B_y \rightarrow \cancel{+i}kE_x = \cancel{i}\omega B_y$

$\hat{z}$: not too useful

$$\vec{B} = \frac{k}{\omega}\left(\hat{z}\times\vec{E}\right)$$

$$\boxed{\vec{B} = \frac{1}{c}\left(\hat{k}\times\vec{E}\right)}$$

(1) $\vec{B}\perp\vec{E}$

(2) $|B| = \frac{|E|}{c}$ since $\hat{k}\perp\vec{E}$

(3) $\vec{B}$ is phase w/ $\vec{E}$

B much less important to effects than E.

(only if particle $\vec{v}$ approaches c since $\vec{F} = q\vec{v}\times\vec{B}$)

Don't need to write all on board

Recall $\nabla \cdot \vec{D} = \rho_{free}$

$= 0$ in regular dielectric

$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$ defn!

Before we used $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$ to obtain

wave eqn with $v = \frac{c}{\sqrt{\epsilon_r}}$ (if $\mu_r = 1$, i.e. nonmagnetic)

i.e. $n = \sqrt{\epsilon_r}$

But does $\vec{D}$ always equal $\epsilon_0 \epsilon_r \vec{E}$? Not necessarily

$\vec{P}$ more fundamental than $\vec{D}$

(and can be complicated $\vec{E} \xrightarrow{\text{causes}} \vec{P}$, $\vec{P} \xrightarrow{\text{affects}} \vec{E}$)

Susceptibility: $\vec{P} = \epsilon_0 \chi \vec{E}$

aka χ_e, because can have a similar mag. χ

how much $\vec{P}$ produced, depends on material

compare $D = \epsilon_0 \epsilon_r E$

$\rightarrow$ In regular materials $\epsilon_r = 1 + \chi$

(often $\vec{P} \propto \vec{E}$, but not always)

if linear $\boxed{\epsilon_r = 1 + \chi}$

convert units

Notes

(1) χ could be matrix: $\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} & & \\ & \chi & \\ & & \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$

must have this if "nonisotropic"

i.e. $\vec{P}$ in different direction than $\vec{E}$

(2) χ could induce nonlinearities

$$P = \epsilon_0 \left(\chi_1 E + \chi_2 E^2 + \dots \right)$$

must have this if "nonlinear"

(3) χ can depend on frequency of light

(4) χ could be complex – phase shift between $\vec{E}$ + $\vec{P}$

We'll worry about (3)+(4) now, (1) later (Ch. 5) and (2) not at all

in grad class, eg Phy 571

χ depends on freq. $\tilde{P}(\omega) = \epsilon_0 \tilde{\chi}(\omega) \tilde{E}(\omega)$

complex: could have phase shift between $\tilde{P}$ and $\tilde{E}$

$\tilde{P} = \tilde{P}_0 e^{i(k\cdot r - \omega t)}$

$\tilde{E} = \tilde{E}_0 e^{i(k\cdot r - \omega t)}$

Back to P + W's wave eq'n

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial J_{free}}{\partial t} + \mu_0 \frac{\partial^2 P}{\partial t^2} - \frac{1}{\epsilon_0} \nabla(\nabla \cdot P)$$

Dielectrics $J_{free} = 0$

Isotropic for now $\nabla \cdot P = 0$

then $\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$ plug in, divide by $e^{i(k\cdot r - \omega t)}$

$$-k^2 \tilde{E}_0 - \frac{1}{c^2}(-\omega^2)\tilde{E}_0 = \mu_0 (-\omega^2)\left(\epsilon_0 \tilde{\chi}(\omega)\right) \tilde{E}_0$$

$1/c^2$

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left(1 + \tilde{\chi}(\omega)\right)}$$

$\frac{\omega}{k} = \frac{c}{\sqrt{1+\chi}}$

Compare: $\epsilon_r = 1 + \chi$

$n = \sqrt{\epsilon_r} = \sqrt{1+\chi}$ from before

Difference: $\tilde{\chi}$ could be imaginary!

define complex index of refraction

$$\boxed{\tilde{N} = \sqrt{1+\tilde{\chi}}}$$

real part of $\tilde{N}$: call "n" like before

imag. part of $\tilde{N}$: call "κ" (kappa, some books "k")

$$\boxed{\tilde{N} = n + i\kappa}$$

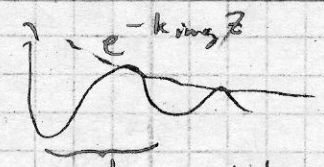
if $\kappa \neq 0 \Rightarrow$ absorption!

proof: $\tilde{k}$ = wave vector $= \frac{\omega}{c}\sqrt{1+\tilde{\chi}} = \frac{\omega}{c}(n + i\kappa)$ also complex!

$\tilde{E} = \tilde{E}_0 e^{i(kz - \omega t)}$ simple case where $k = n\hat{z}$

$= \tilde{E}_0 e^{i[(k_{real} + i k_{imag})z - \omega t]} = \tilde{E}_0 \underbrace{e^{-k_{imag} z}}_{\text{exp. decay! absorption depth}} \underbrace{e^{i(k_{real} z - \omega t)}}_{\text{cosine wavelength}}$

k_{imag} = decay constant of field



$\frac{1}{k_{imag}}$ = distance field goes down by $1/e$

$\alpha = 2 k_{imag}$ often used instead

$\frac{2}{\alpha} = \frac{1}{2 k_{imag}}$ called "absorption depth"

factor of 2: because energy $\sim E^2$

as we'll see later

How to predict $\tilde{N}$ theoretically?

→ Lorentz model of driven damped harmonic oscillator

soln was $\tilde{x}_0 = \frac{qE_0}{m} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}$ $\qquad \tilde{x} = \tilde{x}_0 e^{-i\omega t}$

↓ position of electron

⇓ change to $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

OK because $\cos(\vec{k}\cdot\vec{r} - \omega t)$ doesn't vary much over scale of $\tilde{x}_0$ motion

This oscillatory motion of $\vec{x}$ (electron) produces an electric dipole that changes in time

$\vec{p} = q\vec{x}$ for each electron

dipole moment

$\vec{P} = \frac{\text{total dipoles}}{\text{volume}} = q\vec{x}N$

polarization

↳ # electrons/volume (some books: "n")

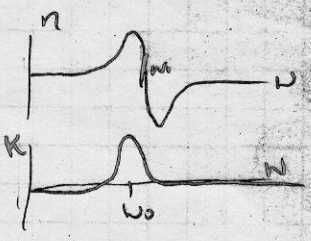
$$\tilde{P} = qN\tilde{x} = (qN)\left(\frac{qE_0}{m}\right)\frac{1}{\omega_0^2 - i\omega\gamma - \omega^2} e^{i(k\cdot\vec{r} - \omega t)} \rightarrow \tilde{E}!$$

$\vec{P} = \epsilon_0 \chi \vec{E}$, so

$$\boxed{\tilde{\chi} = \underbrace{\frac{q^2 N}{\epsilon_0 m}}_{\omega_p^2} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}}$$

with $\omega_p = \sqrt{\frac{q^2 N}{\epsilon_0 m}}$ = "plasma freq"

$$\boxed{\underset{n+iK}{\tilde{N}} = \sqrt{1+\tilde{\chi}}} \longrightarrow \text{can figure out } n \text{ and } K!$$

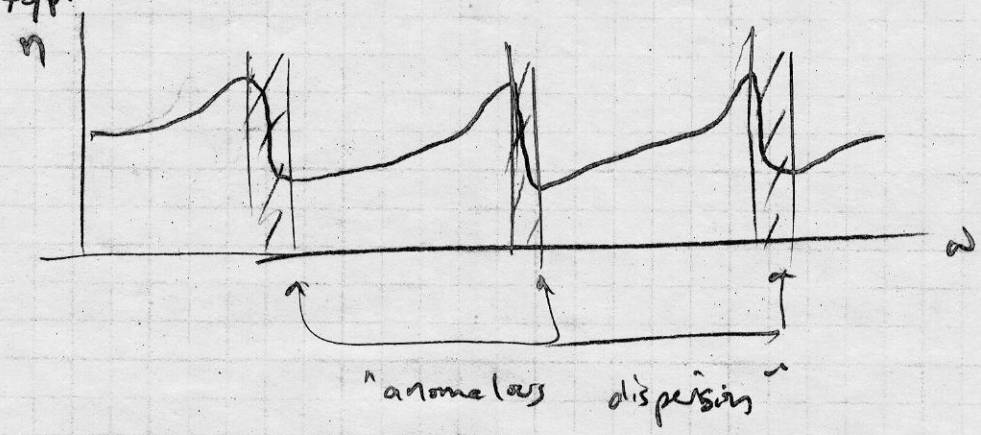


Final modification: if different electrons have eg. different restoring forces, they might have different spring constants → diff ω_0's.
So when we add up all dipoles to get $\tilde{P}$, get fractions w/ diff ω_0

$$\boxed{\tilde{\chi} = \sum_j \frac{f_j \omega_{pj}^2}{\omega_{0j}^2 - i\omega\gamma_j - \omega^2}}$$

f_j = % of "j-type" of electrons
"oscillator strength"

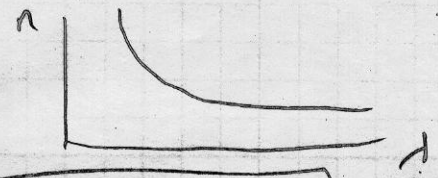
typical material



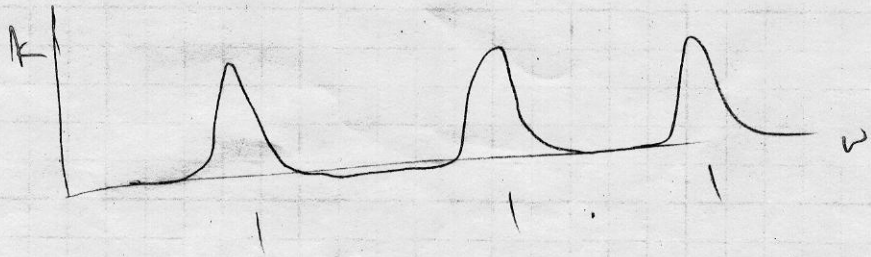
"normal dispersion": n increases w/ ω

i.e. n decreases w/ λ for ω < first ω₀ resonance

for long λ



Hecht picture pg 72
Fig 3.40



different resonances → diff absorptions

QM: comes from energy levels,
not from spring constants

end of lecture 6!
<Whew>

"Normal Dispersion" for transparent materials

From Hecht, pg 72

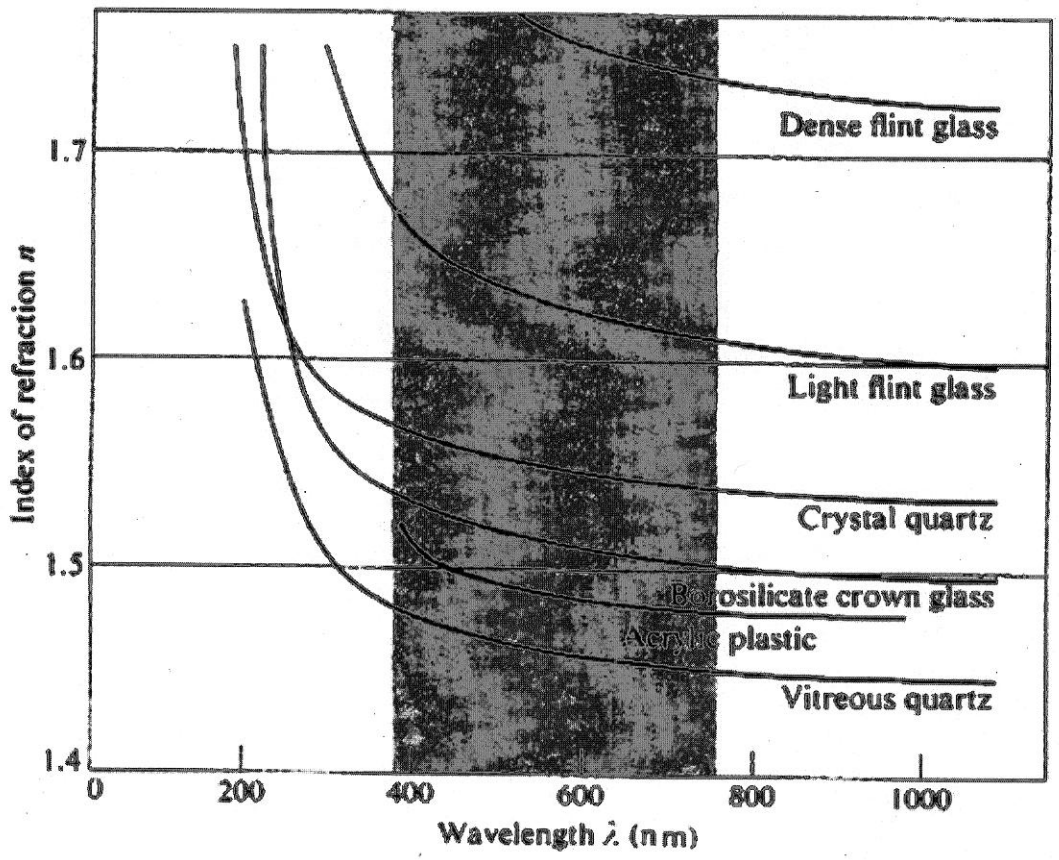


Figure 3.40 The wavelength dependence of the index of refraction for various materials.