

## Lecture 7: Wed, 23 Jan 2008

Reading quizzes: no talking, no looking in your books/notes

Q1. The vector that gives the direction of energy flow through space in an EM wave is the:

- a. E-field
- b. B-field
- c. Clausius vector
- d. current vector
- e. polarization vector
- f. Poynting vector

Q2. In a conductor (metal) the optical properties are described by

- a. a totally real index of refraction
- b. a totally imaginary index of refraction
- c. a complex index of refraction
- d. an index of conduction

Q3. The frequency important to conductors is the

- a. the resonance frequency,  $\omega_0$
- b. the decay frequency,  $\omega_D$
- c. the plasma frequency,  $\omega_p$
- d. the reflection frequency,  $\omega_r$

Conductor let  $\rho = 0$   
 $\nabla \cdot \rho = 0$

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial \mathbf{J}_{free}}{\partial t} + \mu_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} - \frac{1}{\epsilon_0} \nabla(\nabla \cdot \mathbf{P})$$

$$\vec{E} = \vec{E}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$$

assume  $\vec{J}_{free} = \vec{J}_0 e^{i(\mathbf{k} \cdot \mathbf{r} - \omega t)}$

$$\vec{J}_{free} = qN \vec{v}$$

$\frac{\text{charge}}{\text{elec}} \cdot \frac{\text{electron}}{\text{volume}} \cdot \frac{\text{distance}}{\text{time}} = \frac{\text{charge} \cdot \text{time}}{\text{volume} \cdot \text{area}}$

Ohm's Law  $\vec{J} = \sigma \vec{E}$  Ohm's Law (same phase, same exponential)

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \sigma \frac{\partial E}{\partial t}$$

$\Delta$  damping! like Lorentz model  
 from single time derivation

Recall Lorentz model:  $N^2 = 1 + \frac{\omega_p^2}{\omega_0^2 - i\omega\gamma - \omega^2}$   $\omega_p = \sqrt{\frac{Nq^2}{\epsilon_0 m}}$

method 1: solve diff eqn for E. (done in book) use model to relate J and E more carefully than J =  $\sigma E$

method 2: use Lorentz model stuff, w/ no restoring force  $\rightarrow \omega_0 = 0$

$$N^2 = 1 + \frac{\omega_p^2}{-i\omega\gamma - \omega^2}$$

(same result, with  $\sigma = \frac{Nq^2}{m\gamma}$ )  
 $\rightarrow$  can get  $n + ik$  from that

if conductor has bound electrons also, which can absorb also,

$$N^2 = 1 + \frac{\omega_p^2}{\epsilon_0 m} \left[ \frac{f_c}{-i\omega\gamma_c - \omega^2} + \sum_j \frac{f_j}{\omega_{0j}^2 - \omega^2 + i\gamma_j \omega} \right]$$

$N = \# \text{ atoms} / \text{vol.}$

$f_c = \# \text{ cond. electrons} / \text{atom}$

$f_j = \# \text{ bound electrons} / \text{atom}$  at least res. freq.

instead of  $\# \text{ electrons} / \text{volume}$

What does  $\omega_p$  represent?  Then kill  $\vec{E} \rightarrow$  what happens?

PIU Derivation = skip!  
 $\nabla^2 E = \mu_0 \frac{\partial J}{\partial t} = \mu_0 \frac{\partial J}{\partial t}$

$$\vec{J}_{\text{free}} = q N \vec{v}_{\text{micro}}$$

$$\sum F = m a$$

$$m \dot{v} = q E - m \gamma v_{\text{micro}}$$

Steady state:  $\dot{v} = 0$

$$q E = m \gamma v_{\text{micro}}$$

$$\vec{J}_{\text{free}} = q N \left( \frac{q E}{m \gamma} \right)$$

$$\vec{J} = \left( \frac{q^2 N}{m \gamma} \right) E$$

$$\vec{J} = \sigma \vec{E} \rightarrow \text{Ohm's law w/ } \sigma = \frac{q^2 N}{m \gamma} \quad \text{only true for DC}$$

only true when  $\dot{v} = 0$ ?

general  $m \dot{v} = q E - m \gamma v$

$$m(-i\omega) \tilde{v} = q \tilde{E} - m \gamma \tilde{v}$$

$$\tilde{v} = \frac{q \tilde{E}}{m} \frac{1}{\gamma - i\omega}$$

Then  $\vec{J} = q N \tilde{v} = \frac{q^2 N \tilde{E}}{m} \frac{1}{\gamma - i\omega}$

Correspondence between  $\omega$  &  $\gamma$  &  $\mu_0$   
 $\omega \neq \gamma$ :  $\dot{v} \neq 0$   
 $\rightarrow$  auto only true if large damping

$$(ik)^2 \tilde{v} / \mu_0 \tilde{E} = \mu_0 \frac{q^2 N}{m} \frac{1}{\gamma - i\omega} \tilde{E}$$

$$-k^2 + \frac{1}{c^2} \omega^2 = \mu_0 \frac{q^2 N}{m} \left( \frac{1}{\gamma - i\omega} \right) (i\omega)$$

$$k^2 = \frac{\omega^2}{c^2} + \left( \frac{\mu_0 N q^2}{m} \right) \frac{i\omega}{\gamma - i\omega} \frac{1}{i}$$

$$\frac{\omega}{-i\gamma - \omega}$$

$$k^2 = \frac{\omega^2}{c^2} - \left( \frac{\mu_0 N q^2}{m} \right) \frac{\omega}{i\gamma + \omega}$$

"Good conductor"  $\delta = \text{small}, \ll \omega$

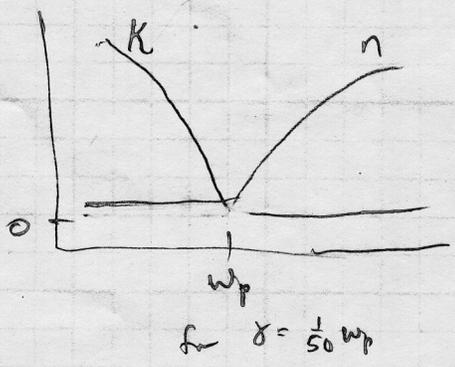
then  $n^2 = j \ll \frac{\omega_p^2}{\omega^2}$

Hecht:  $\omega_p$  serves as a critical value below which the index is complex and the penetrating wave drops off exponentially.

$\omega > \omega_p$ :  $n = \text{real}$ , absorption = small, conductor = transparent  
↳ less than 1

$\omega < \omega_p$ :  $n = \text{complex}$ , absorption is good, conductor reflects (I think)

Book figure pg 42 Fig 2.4b



$\frac{1}{\omega_p} = \frac{1}{c} \sqrt{\epsilon_0 \mu_0}$   
 by fact

$\frac{1}{\omega_p} = \frac{1}{c} \sqrt{\epsilon_0 \mu_0}$

# Reflectance vs wavelength for some metals

From Hecht

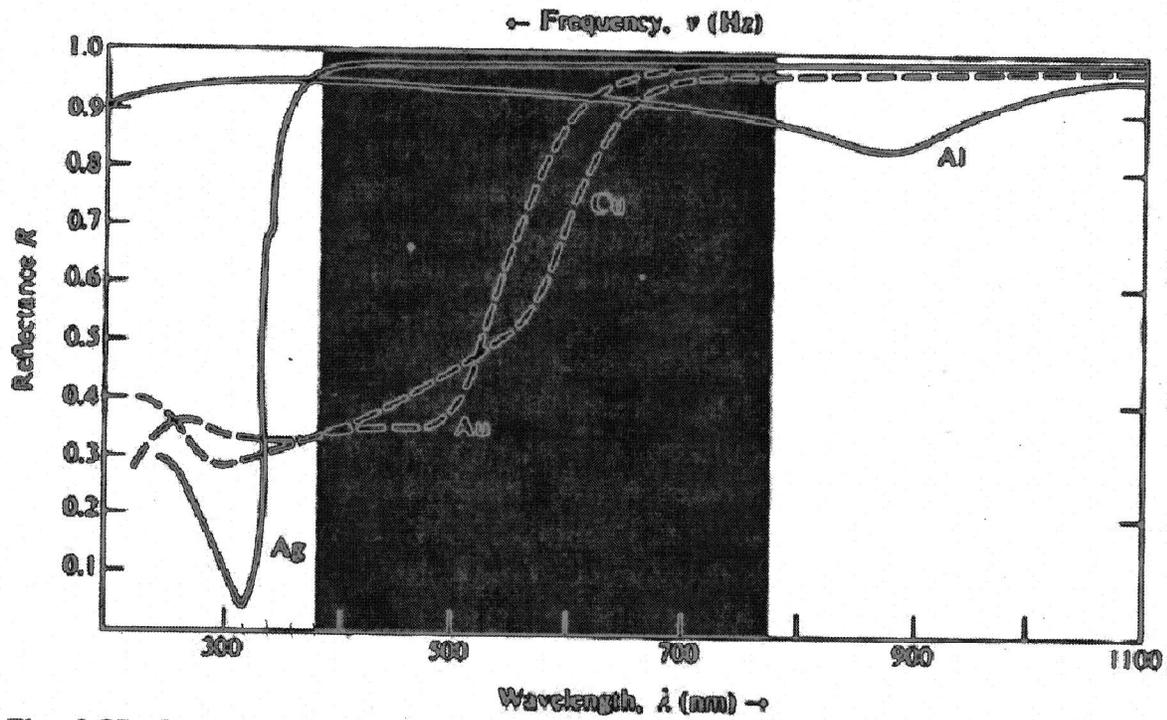


Fig. 4.37 Reflectance versus wavelength for silver, gold, copper and aluminum.

# Energy + Power in EM fields

Appendix (also in Griffiths): (I won't derive)

energy stored in electric field:  $U = \frac{\epsilon_0}{2} \int E^2 dv$

" " " magnetic "  $U = \frac{1}{2\mu_0} \int B^2 dv$

= how much energy it took to assemble the charges /  
 set up the currents producing the fields

energy density  $\frac{U}{\text{volume}}$

$u_{\text{field}} = \frac{\epsilon_0}{2} E^2 + \frac{B^2}{2\mu_0}$

lower case

energy flow = ~~power~~ (we'll actually do  $\frac{\text{power}}{\text{volume}}$ )

start with Maxwell Eqns 2 + 4

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$$

$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$   
 dot  $\vec{B}$

$\nabla \times \left( \frac{\mathbf{B}}{\mu_0} \right) = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \frac{\partial \mathbf{P}}{\partial t} + \mathbf{J}_f$  (book's version)  
 when  $\rho_{\text{ext}} = 0$   
 $M = 0$

$\mathbf{B} \cdot (\nabla \times \mathbf{E}) = \underbrace{\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}}_{\frac{1}{2} \frac{\partial}{\partial t} (B^2)}$   
 vector idem  
 $\nabla \cdot (\mathbf{E} \times \mathbf{B}) + \mathbf{E} \cdot (\nabla \times \mathbf{B})$

$\frac{1}{\mu_0} \mathbf{E} \cdot (\nabla \times \mathbf{B}) = \epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{E} \cdot (\mathbf{J}_f + \mathbf{J}_p)$   
 $\frac{1}{2} \frac{\partial}{\partial t} (E^2)$

$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = -\frac{1}{2} \frac{\partial}{\partial t} (B^2) - \nabla \cdot (\mathbf{E} \times \mathbf{B})$

$\frac{1}{\mu_0} \left( -\frac{1}{2} \frac{\partial}{\partial t} (B^2) - \nabla \cdot (\mathbf{E} \times \mathbf{B}) \right) = \epsilon_0 \frac{1}{2} \frac{\partial}{\partial t} (E^2) + \mathbf{E} \cdot (\mathbf{J}_f + \mathbf{J}_p)$

Poynting Theorem

continuity =  $\frac{\text{power}}{\text{volume}}$

$\nabla \cdot \left( \mathbf{E} \times \frac{\mathbf{B}}{\mu_0} \right) + \frac{\partial}{\partial t} \left( \frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0} \right) = -\mathbf{E} \cdot (\mathbf{J}_f + \mathbf{J}_p)$

$\vec{S}$        $u_{\text{field}}$        $-\frac{\partial u_{\text{medium}}}{\partial t}$

$\nabla \cdot \vec{S} + \frac{\partial u_{\text{field}}}{\partial t} = -\frac{\partial u_{\text{medium}}}{\partial t}$

power flow!  
 energy flow out of region  
 change in energy of EM fields  
 work done on charges per time

EX: if medium is general, if need boundary about magnetic material

$$\nabla \cdot \vec{S} + \frac{\partial u_{free}}{\partial t} = - \frac{\partial u_{medium}}{\partial t}$$

integrate:

$$\int (\nabla \cdot \vec{S}) dv + \frac{dU_{free}}{dt} = - \frac{dU_{medium}}{dt}$$

$$\frac{dU_{free}}{dt} = - \text{work done} - \text{energy transported away}$$

Poynting vector  $\vec{S}$ :

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B}$$

$\vec{E} \times \vec{B}$  = direction of energy flow  
fits with  $\hat{E} \times \hat{B} = \hat{k}$  picture

plane wave:  $\vec{B} \perp \vec{E}$   
 $\vec{E}$  and  $\vec{B}$  in phase

$$S(t) = \frac{1}{\mu_0} (E \cos \omega t) \left( \frac{E}{c} \cos \omega t \right)$$

$$S(t) = \frac{E^2}{\mu_0 c} \cos^2(\omega t)$$

$$\text{average } \langle \cos^2 \rangle = \frac{1}{2}$$

$$\langle S \rangle = \frac{1}{2} \frac{E^2}{\mu_0 c}$$

= intensity of light  
 $\frac{\text{power}}{\text{area}}$

$$eV E^2$$

end of Ch. 21

(skip last section)

in matter:  $\frac{1}{2} (E) \left( \frac{B}{\mu} \right) = \frac{1}{2} (E) \left( \frac{E}{\mu v} \right)$   $c \rightarrow \frac{1}{\mu_0}$   
 $\mu \neq \mu_0$   $v$  index  $\rightarrow n$  in numerator  
leads to same as back?  $\frac{1}{2} n^2 \epsilon_0 E^2$

**Lecture 8: Fri, 25 Jan 2008**

Reading quizzes: no talking, no looking in your books/notes

Q1. The plane containing the incoming, reflected and transmitted k-vectors is the plane of \_\_\_\_\_

- a. incidence
- b. intersection
- c. reflection
- d. refraction
- e. transmission
- f. reflectotransmission

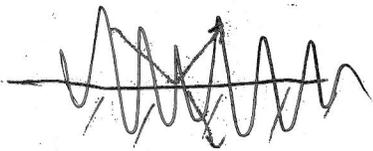
Q2. To represent the components of E (polarization) with respect to the above plane we use the symbols (choose 2):  p, q, r,  s, t, u, v

Q3. The Fresnel coefficients,  $r$  and  $t$ :  $n_2 \neq R, T$

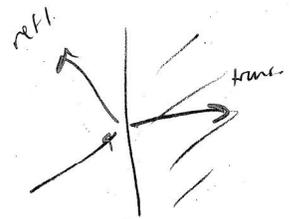
- a. describe how much electric field is reflected and transmitted at an interface
- b. describe how much intensity is reflected and transmitted at an interface
- c. must add up to 1
- d. more than one of the above
- e. none of the above

Almost ready to tackle the big problem

day 8



how much reflected?  
how much transmitted?



Reminder of boundary conditions:

(i)  $\epsilon_1 E_{1\perp} = \epsilon_2 E_{2\perp}$  (from  $\nabla \cdot \vec{D} = \rho_{free}$ , if no surface charge)

(ii)  $\vec{E}_{1\parallel} = \vec{E}_{2\parallel}$  (from  $\nabla \times \vec{E} = -\partial \vec{B} / \partial t$ )

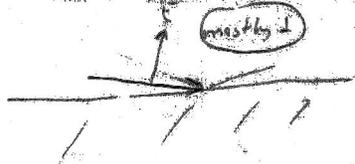
(iii)  $B_{1\perp} = B_{2\perp}$  (from  $\nabla \cdot \vec{B} = 0$ )

(iv)  $\frac{B_{1\parallel}}{\mu_1} = \frac{B_{2\parallel}}{\mu_2}$  (from  $\nabla \times \vec{H} = \vec{J}_{free} + \frac{\partial \vec{D}}{\partial t}$  if no surface current)

For us,  $\mu_1 = \mu_2 = \mu_0$  so  $B_{1\parallel} = B_{2\parallel}$

Complication: polarization will matter!

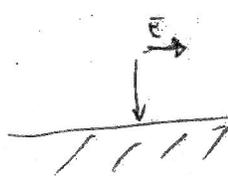
"p polarization" p = parallel to plane of incidence



"s polarization" s = perpendicular to plane of incidence

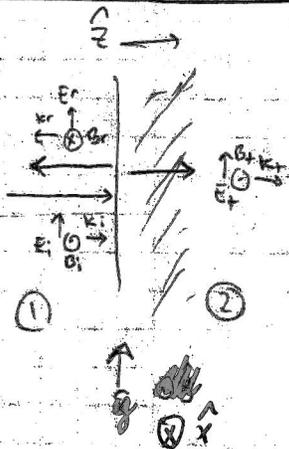


Getting our feet wet... let's tackle  $\theta = 90^\circ$  "Normal incidence"



then  $E$  is all  $\parallel$ , (s + p polarization indistinguishable; there is no "plane of incidence")

# Reflection & Transmission at Normal Incidence



- Assume all  $\vec{E}$ 's in  $\hat{y}$  direction  
 ↳ if not, they'd come out negative in the end
- Directions of  $\vec{B}$  given by  $\vec{E} \times \vec{B} = \hat{k}$
- each wave has form  $\vec{E}_0 e^{i(kz - \omega t)}$
- mag of  $\vec{B}$  given by  $\frac{E}{v}$

Write down waves:

incident

$$\vec{E}_I = \vec{E}_{0I} e^{i(k_1 z - \omega t)} \hat{y}$$

$$\vec{B}_I = \frac{1}{v_1} \vec{E}_{0I} e^{i(k_1 z - \omega t)} \hat{x}$$

reflected

$$\vec{E}_R = \vec{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{y}$$

$$\vec{B}_R = \frac{1}{v_1} \vec{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{x}$$

transmitted

$$\vec{E}_T = \vec{E}_{0T} e^{i(k_2 z - \omega t)} \hat{y}$$

$$\vec{B}_T = \frac{1}{v_2} \vec{E}_{0T} e^{i(k_2 z - \omega t)} \hat{x}$$

plug into boundary eqns, set  $z=0$ , do some algebra, then we're done!

~~at  $z=0$ :  $\vec{E}_1 \parallel = \vec{E}_2 \parallel$  and  $\vec{B}_1 \perp = \vec{B}_2 \perp$~~

continuity of  $\vec{E}$  at  $z=0$ :  $\vec{E} = \vec{E}_{0I} e^{i(k_1 z - \omega t)} \hat{y} + \vec{E}_{0R} e^{i(-k_1 z - \omega t)} \hat{y}$

continuity of  $\vec{B}$  at  $z=0$ :  $\vec{B} = \vec{E}_{0T} e^{i(k_2 z - \omega t)} \hat{x}$

at  $z=0$ :

$E_{1\parallel} = E_{2\parallel}$ :  $E_{0I} e^{-i\omega t} + E_{0R} e^{-i\omega t} = E_{0T} e^{-i\omega t} \Rightarrow \boxed{E_{0I} + E_{0R} = E_{0T}}$

at  $z=0$ :

$B_{1\perp} = B_{2\perp}$ :  $\frac{E_{0I}}{v_1} e^{-i\omega t} \hat{x} + \frac{E_{0R}}{v_1} e^{-i\omega t} \hat{x} = \frac{E_{0T}}{v_2} e^{-i\omega t} \hat{x} \Rightarrow \boxed{\frac{E_{0I}}{v_1} + \frac{E_{0R}}{v_1} = \frac{E_{0T}}{v_2}}$

$\boxed{E_{0I} + E_{0R} = \frac{v_1}{v_2} E_{0T}}$

Solve for  $E_{0T}$ , plug in, → solve for  $E_{0R}$  in terms of  $E_{0I}$

let  $\beta = \frac{v_1}{v_2} = \frac{n_2}{n_1}$

$$I - R = \beta (E + R)$$

$$I - R = \beta I + \beta R$$

$$I - \beta I = R + \beta R$$

$$I(1 - \beta) = R(1 + \beta)$$

$$\tilde{E}_{0R} = \frac{1 - \beta}{1 + \beta} \tilde{E}_{0I} = \frac{n_1 - n_2}{n_1 + n_2} \tilde{E}_{0I} = \frac{v_2 - v_1}{v_2 + v_1} \tilde{E}_{0I}$$

→ 180° phase shift if  $v_2 < v_1$ , just like ropes!

Similarly

$$\tilde{E}_{0T} = \frac{2}{1 + \beta} \tilde{E}_{0I} = \frac{2n_1}{n_1 + n_2} \tilde{E}_{0I} = \frac{2v_2}{v_2 + v_1} \tilde{E}_{0I}$$

if  $\mu_1 = \mu_2$  (since they both likely  $\approx \mu_0$ )

then  $\tilde{E}_{0R} = \frac{1 - \frac{v_2}{v_1}}{1 + \frac{v_2}{v_1}} \tilde{E}_{0I}$

$$\tilde{E}_{0R} = \frac{v_2 - v_1}{v_2 + v_1} \tilde{E}_{0I}$$

and  $\tilde{E}_{0T} = \frac{2v_2}{v_2 + v_1} \tilde{E}_{0I}$

identical to string waves!

in particular, get phase shift of 180° if  $v_2 < v_1$  ( $n_2 > n_1$ )

↳ then  $E_r = \downarrow$ , not up!

since  $\frac{n_1}{n_2} = \frac{v_2}{v_1}$ , it's also

$$\tilde{E}_{0R} = \left| \frac{n_1 - n_2}{n_1 + n_2} \right| \tilde{E}_{0I}$$

→ the net amplitudes must be positive

$$\tilde{E}_{0T} = \frac{2n_1}{n_1 + n_2} \tilde{E}_{0I}$$

$\frac{n_1}{n_2} = \frac{v_2}{v_1}$  done before

since Power  $= \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H} \}$

$$R = \frac{P_R}{P_I}$$

$$\Rightarrow R = \left| \frac{n_1 - n_2}{n_1 + n_2} \right|^2$$

reflectance

$$T = \frac{P_T}{P_I}$$

$$\Rightarrow T = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

$$T = \frac{4n_1 n_2}{(n_1 + n_2)^2}$$

transmittance  
P/W = "transmission"

can show  $R + T = 1$ ; conservation of energy!

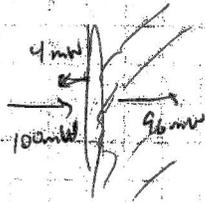
$$\langle S \rangle = \frac{1}{2} \text{Re} \{ \mathbf{E} \times \mathbf{H} \}$$

STANFORD

Example: air/glass ( $n=1.5$ ), 100mW laser  
( $n=1$ )

$$R = \left( \frac{1-1.5}{2.5} \right)^2 = 4\%$$

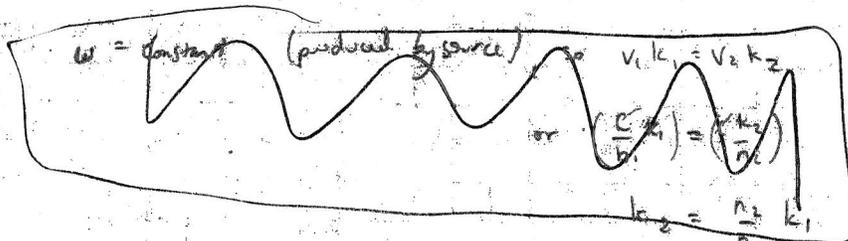
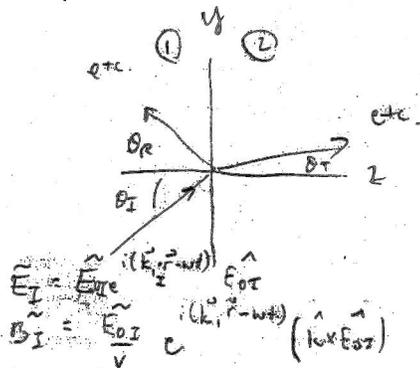
$$T = \frac{4 \cdot 1.5}{2.5^2} = 96\%$$



optics rule: lose 4% at every air/glass surface

→ might have to worry (safety) about  
reflections off of glass.

At an angle - not assume any polarization yet



Same magnitude as  $k_{i,z}$  but different direction

B.C.  $(E_{1z} = E_{2z})_{z=0}$   $(B_{1y} = B_{2y})_{z=0}$

at  $z=0$

$$E_{0i} e^{i(k_{ix}x + k_{iy}y - \omega t)} + E_{0r} e^{i(k_{rx}x + k_{ry}y - \omega t)} = E_{0t} e^{i(k_{tx}x + k_{ty}y - \omega t)}$$

$$E_{0i} e^{i(k_{ix}x + k_{iy}y - \omega t)} + E_{0r} e^{i(k_{rx}x + k_{ry}y - \omega t)} = E_{0t} e^{i(k_{tx}x + k_{ty}y - \omega t)}$$

must all be equal  
 else get things like  $(x=0) - E_{iz} + E_{ir} = E_{it}$   
 $(y=0) - E_{iz} \cos(k_{iy}y) + E_{ir} \cos(k_{ry}y) = E_{it} \cos(k_{ty}y)$   
 integrating to be equal

$$k_{ix}x + k_{iy}y = k_{tx}x + k_{ty}y$$

$$k_{ix}x + k_{iy}y = k_{tx}x + k_{ty}y$$

Similar reasoning, must hold separately for x + y

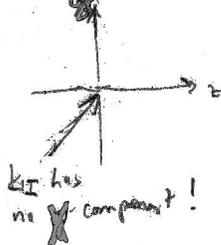
$$k_{ix}x = k_{tx}x$$

$$k_{iy}y = k_{ty}y$$

also  $k_{ix}x = k_{tx}x$

also  $k_{iy}y = k_{ty}y$

orient x-axis as before



but  $k_{iy} = 0!$

Therefore

$$\left. \begin{aligned} k_{iy} &= 0 \\ k_{ty} &= 0 \end{aligned} \right\}$$

1st Law!  
 entire thing takes place in  $yz$  plane!  
 "plane of incidence"