day 14 pg 1

(quizzes)

Crystals - ordered structure of atoms

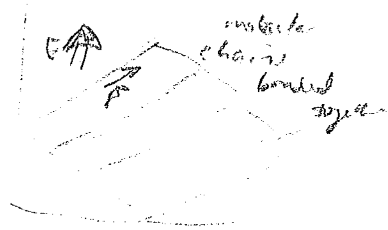
chap 5 - can't have preferred direction

"non isotropic"

$$\vec{P} = \epsilon_0 \overleftrightarrow{\chi} \vec{E}$$

$$\begin{pmatrix} P \end{pmatrix} = \epsilon_0 \underbrace{\begin{pmatrix} \chi \end{pmatrix}}_{\begin{pmatrix} \chi_{xx} & \chi_{xy} & \chi_{xz} \\ \chi_{yx} & \chi_{yy} & \chi_{yz} \\ \chi_{zx} & \chi_{zy} & \chi_{zz} \end{pmatrix}} \begin{pmatrix} E \end{pmatrix}$$

example



Simplify: χ components = real, χ = symmetric } if no absorption and no "optical activity"

~~Simplify: "ordinary crystals": χ = symmetric~~

then (matrix theorem) you can rotate coord. axes to get

ie. "$\hat{x}$" = $\begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$ in above picture

$$\begin{pmatrix} \chi_{xx} & & \\ & \chi_{yy} & \\ & & \chi_{zz} \end{pmatrix}$$ only

proved in Appendix 5.A

then $$\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} \chi_{xx} & & \\ & \chi_{yy} & \\ & & \chi_{zz} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$$

gives:
$$P_x = \epsilon_0 \chi_{xx} E_x$$
$$P_y = \epsilon_0 \chi_{yy} E_y$$
$$P_z = \epsilon_0 \chi_{zz} E_z$$
} pretty similar to isotropic case

Next lecture [Simplify (some crystals): $\chi_{xx} = \chi_{yy}$, χ_{zz}: different

↓

"uniaxial"

quartz, sapphire

Back to wave eqn (1.41), have no J_{free}

$$\nabla^2 E - \mu_0\epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \overbrace{\cancel{\frac{\partial J_{free}}{\partial t}}}^{=0} + \mu_0 \frac{\partial^2 P}{\partial t^2} - \frac{1}{\epsilon_0}\nabla(\nabla\cdot P)$$

Plane waves $\vec{E} = \vec{E}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}$, similar for $\vec{B}$ and $\vec{P}$

Maxwell

$\nabla\cdot B = 0 \Rightarrow$ last time, $\vec{k}\cdot\vec{B} = 0$ $\quad B \perp E$

$\nabla\cdot\vec{E} = \rho/\epsilon_0 \Rightarrow$ last time $= 0$ so $\vec{k}\cdot\vec{E} = 0$

now not! $\vec{D}$ not $\parallel \vec{E}$ so even if $\rho_{free} = 0 \Rightarrow \nabla\cdot D = 0$ doesn't force $\nabla\cdot E = 0$

$\nabla\cdot\vec{D} = \rho_{free} \longrightarrow \nabla\cdot D = 0$

instead $\Rightarrow \nabla\cdot(\epsilon_0 E + P) = 0$

$\vec{k}\cdot(\epsilon_0\vec{E} + \vec{P}) = 0$

if $\vec{P}$ not in $\vec{E}$ direction, then $\vec{k}\cdot\vec{E} \neq 0$

$\underline{\underline{E \text{ not} \perp \vec{k}}}$! (but $\vec{D}$ is)

but $\vec{S} = \vec{E}\times\frac{\vec{B}}{\mu_0}$ must be $\perp \vec{E}$

So $\vec{S}$ $\underline{\underline{\text{not} \parallel \vec{k}}}$

flow of energy + direction of wave propagation not the same!

Goal: solve for n in specific direction of travel ($\vec{k}$)

Wave eqn:

$$-k^2 E - \mu_0\epsilon_0(-\omega^2 E) = \mu_0(-\omega^2)P - \frac{1}{\epsilon_0}\nabla(\underbrace{-\nabla\cdot\epsilon_0 E}_{\text{was } \nabla\cdot P})$$

$$-k^2 E + \omega^2\mu_0(\epsilon_0 E + P) = \begin{matrix} + ik(ik\cdot E) \\ -\vec{k}(\vec{k}\cdot E)\end{matrix}$$

check signs! → see my 11/26/12

$$k^2\vec{E} - \frac{\omega^2}{c^2}(\vec{E} + \vec{\chi}\vec{E}) = \vec{k}(\vec{k}\cdot\vec{E})$$

$$k^2\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} - \frac{\omega^2}{c^2}\left[\begin{pmatrix}E_x\\E_y\\E_z\end{pmatrix} + \begin{pmatrix}\chi_{xx}E_x\\ \chi_{yy}E_y\\ \chi_{zz}E_z\end{pmatrix}\right] = \begin{pmatrix}k_x\\k_y\\k_z\end{pmatrix}(\vec{k}\cdot\vec{E})$$

$$\left[k^2 - \frac{\omega^2}{c^2}(1+\chi_{xx})\right]E_x = k_x(\vec{k}\cdot\vec{E})$$
$$\left[k^2 - \frac{\omega^2}{c^2}(1+\chi_{yy})\right]E_y = k_y(\vec{k}\cdot\vec{E})$$
$$\left[k^2 - \frac{\omega^2}{c^2}(1+\chi_{zz})\right]E_z = k_z(\vec{k}\cdot\vec{E})$$

→ For each eqn, divide by $\left[k^2 - \frac{\omega^2}{c^2}(1+\chi)\right]$

$$\begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix} = \vec{k}\cdot\vec{E} \begin{pmatrix} k_x / k^2 - \frac{\omega^2}{c^2}(1+\chi_x) \\ k_y / k^2 - \frac{\omega^2}{c^2}(1+\chi_y) \\ k_z / k^2 - \frac{\omega^2}{c^2}(1+\chi_z) \end{pmatrix}$$

If eg $k^2 - \frac{\omega^2}{c^2}(1+\chi_x) = 0$ then mathematical problem ($\div 0$) there. We'll return to that case in an example

multiply both sides by $(k_x \; k_y \; k_z)$

$$\vec{k}\cdot\vec{E} = \vec{k}\cdot\vec{E}\left[\frac{k_x^2}{k^2 - \frac{\omega^2}{c^2}(1+\chi_x)} + \frac{k_y^2}{k^2 - \frac{\omega^2}{c^2}(1+\chi_y)} + \frac{k_z^2}{k^2 - \frac{\omega^2}{c^2}(1+\chi_z)}\right]$$

multiply both sides by $\frac{1}{c^2/\omega^2} \cdot \frac{1}{k^2}$

$$\frac{1}{k^2 c^2/\omega^2} = \frac{u_x^2}{k^2 c^2/\omega^2 - (1+\chi_x)} + \frac{u_y^2}{\frac{k^2c^2}{\omega^2} - (1+\chi_y)} + \frac{u_z^2}{\frac{k^2c^2}{\omega^2} - (1+\chi_z)}$$

with $u_x = \frac{k_x}{k}$, $u_y = k_y/k$, $u_z = k_z/k$ } $\vec{u}$ = unit vector in $\vec{k}$ direction

Substitute in $n^2 = \frac{k^2c^2}{\omega^2}$, def. $n_x = \sqrt{1+\chi_x}$, $n_y = \sqrt{1+\chi_y}$, $n_z = \sqrt{1+\chi_z}$ } as makes sense $n = \begin{pmatrix} n_x & & \\ & n_y & \\ & & n_z \end{pmatrix}$

$$\boxed{\frac{1}{n^2} = \frac{u_x^2}{n^2 - n_x^2} + \frac{u_y^2}{n^2 - n_y^2} + \frac{u_z^2}{n^2 - n_z^2}}$$

Fresnel's Eqn for crystal optics

Suppose you know direction of $\vec{k}$ (hence u_x, u_y, u_z)
and n_x, n_y, n_z (index of refraction along principal axes)

→ then you can use this eqn to solve for n in your k-direction
(then you could get magnitude of k from $\frac{\omega}{k} = \frac{c}{n}$)

Awfully close to eqn for ellipsoid if multiply by n^2 and $\frac{1}{\frac{1}{n^2}}$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

$$a = 1 - \left(\frac{n_x}{n}\right)^2 \quad \text{etc}$$

~~Example~~

Say $n = \begin{pmatrix} 1.2 & & \\ & 1.3 & \\ & & 1.4 \end{pmatrix}$ $\quad n_x = 1.2$, $n_y = 1.3$, $n_z = 1.4$

and $\vec{k}$ in $(1,1,0)$ direction $\quad u_x = 1/\sqrt{2}$, $u_y = 1/\sqrt{2}$

Fresnel Eqn: $\frac{1}{n^2} = \frac{1/2}{n^2 - 1.2^2} + \frac{1/2}{n^2 - 1.3^2} + 0$

Algebraically: multiply by $(n^2 - 1.2^2)(n^2 - 1.5^2)$

Solve for n^2. Not too bad

Or... Mathematica

$n = 1.247$

Example 1

What if $\vec{k}$ in $(1,1,1)$ direction? $\quad u_x = 1/\sqrt{3}$, $u_y = 1/\sqrt{3}$, $u_z = 1/\sqrt{3}$

$$\frac{1}{n^2} = \frac{1/3}{n^2 - 1.2^2} + \frac{1/3}{n^2 - 1.3^2} + \frac{1/3}{n^2 - 1.4^2}$$

Algebra much worse!
quadratic formula with $n^2 = $ "x"
See textbook

Or ... Mathematica!

$n = \begin{cases} 1.23836 \\ 1.35397 \end{cases}$

Two n's! For two polarizations.
Figuring out details on what polar. dir $\leftrightarrow$ what n isn't in book
Relates to eigenvalue-like problem
For given n, what are E_x, E_y, E_z?

Example 2

$u = (1,0,0)$

$$\frac{1}{n^2} = \frac{1}{n^2 - 1.2}\left(\frac{0}{?} + \frac{0}{?}\right) \quad \text{Problem!}$$

$k^2 - \frac{\omega^2}{c^2}(1+\chi_x) = 0$

For this case, so our division 2 pages ago was illegal

— One approach: limit

$$u = \frac{1}{\sqrt{100^2+1^2+1^2}}(100,1,1)$$

$$\rightarrow n = \begin{cases} 1.3 & \text{if } E \text{ is in } y \text{ direction!} \\ 1.4 & \text{if } E \text{ is in } z \text{ direction!} \end{cases}$$

<u>not</u> $n = 1.2$

because χ_x, χ_y, χ_z refer to the direction of the field, not direction of $\vec{k}$!

— Another approach: Eqn can only be true if one of the denominators $\frac{0}{?}$ is 0

$\rightarrow n = 1.3$
or $n = 1.4$

— Another approach: go back to 3 eqns w/ k_x, k_y, k_z and substitute in them. (best!)

For example, since $k_z = 0$, 3^{rd} equation becomes

$$\left(k^2 - \frac{\omega^2}{c^2}(1+\chi_{zz})\right)E_z = 0$$

$$k^2 = \frac{\omega^2}{c^2}(1+\chi_{zz})$$

$$\therefore \frac{kc}{\omega} = \sqrt{1+\chi_{zz}}$$

$\underline{n = n_z}$ is a solution

— Book's approach: (I don't like) use quadratic formula stuff they derive

— <u>Summary</u>: remember if one of k components is 0, then $n = n_{\text{that direction}}$ is a valid value!

Example 3 Same $n = \begin{pmatrix} 1.2 & 1.3 & 1.4 \end{pmatrix}$

k in $(1,1,0)$ direction $u = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

— Answer: $n = n_z$ is one answer $= \underline{1.4}$

Other answer from

$$\frac{1}{n^2} = \frac{1/2}{n^2 - 1.2^2} \quad \frac{1/2}{n^2 - 1.3^2}$$

(Mathematica)

$$n = \underline{1.247}$$

— Method 2: limit ... $u = \frac{1}{\sqrt{20001}} \begin{pmatrix} 100 \\ 100 \\ 1 \end{pmatrix}$

(Mathematica) $<$ 1.247, 1.4

Example 4 k in (.747, 0, .665) direction

Formula for "magic direction" in book

$$\cos\theta = \pm \frac{n_x}{n_y}\sqrt{\frac{n_z^2-n_y^2}{n_z^2-n_x^2}} = \pm \frac{1.2}{1.3}\sqrt{\frac{1.4^2-1.3^2}{1.4^2-1.2^2}}$$

skip formula for now

$$\cos\theta = \pm .6651$$

$$\theta = 48.3^\circ, \ 131.7^\circ$$

$$\phi = 0$$

$u_x = \sin\theta\cos\phi$

$u_y = \sin\theta\sin\phi$

$u_z = \cos\theta$

$$\begin{pmatrix} .7467 \\ 0 \\ .6651 \end{pmatrix} \text{ or } \begin{pmatrix} .7467 \\ 0 \\ -.6651 \end{pmatrix}$$

What is n?

one value: $\underline{n = n_y = 1.3}$ is valid

Other one: $$\frac{1}{n^2} = \frac{.7467^2}{n^2 - 1.2^2} + \frac{.6651^2}{n^2 - 1.4^2}$$

$n = 1.3$ ✓ same value as 1st one

(other large n = numerical artifact)

for these two magic directions, $\underline{\text{both polarizations have}}$ $\underline{\text{the same } n}$!