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Section 10.5 Fraunhofer Approx.

$$e^{i\frac{k}{2d}(x'^2+y'^2)} \simeq 1$$

True if $d \to \infty$ "far field"

What about $e^{-i\frac{k}{d}(xx'+yy')}$ term? Not also $=1$?

→ Not necessarily! x' and y' limited by size of aperture

x and y have no such limit.

Result

$$E \simeq -\frac{i}{\lambda d}\, e^{ikd}\, e^{i\frac{k}{2d}(x^2+y^2)} \iint_{\text{aperture}} E(x',y')\, e^{-i\frac{k}{d}(xx'+yy')}\, dx'dy'$$

define $\boxed{k_x = \frac{kx}{d}}$ $\boxed{k_y = \frac{ky}{d}}$ $k = \frac{2\pi}{\lambda}$ normal wavevector

units of wavevector (m^{-1}), but Not a wavevector

$$\underbrace{\iint E(x',y')\, e^{-ik_x x'} e^{-ik_y y'}\, dx'dy'}$$

2D Fourier Transform of aperture function! (w/o 2π)

(or inverse FT to be technical)

x' like t

k_x like ω

We mainly want intensity, $I \propto |E|^2$. Stuff in front of integral except $\frac{1}{\lambda d}$ goes away

overall brightness/darkness Not pattern

Also Consider $\left|\int f(x) e^{ikx} dx\right|^2$ vs $\left|\int f(x) e^{-ikx}\right|^2$

They are equal! Because $|\,|^2 = [\;] \times [\;]^*$

And since $f(x)$ is real, each $[\;]$ CC of the other

Therefore FT vs IFT doesn't effect intensity

Important! ☆ $\boxed{I = I_0 \times \left|\text{2D F.T. of aperture function}\right|^2}$

↑ can even lump $\frac{1}{\sqrt{2\pi}}$ factors into I_0

Often aperture function $\begin{cases} 1 & \text{inside aperture} \\ 0 & \text{outside aperture} \end{cases}$

can be otherwise, e.g. filter over part of aperture

2D Fourier Transforms

often our aperture function will be separable

$$E(x', y') = E(x')\, E(y')$$

then

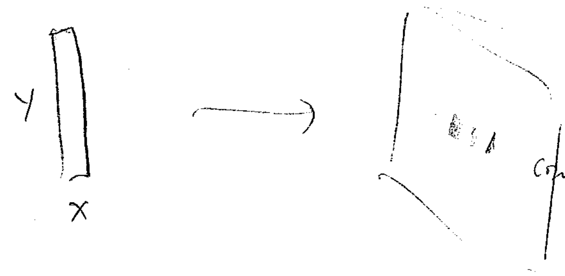
$$\iint E(x', y')\, e^{ik_x x'}\, e^{ik_y y'}\, dx'\, dy'$$

$$= \int E(x')\, e^{ik_x x'}\, dx' \int E(y')\, e^{ik_y y'}\, dy'$$

$$= FT(x \text{ function}) \cdot FT(y \text{ function})$$

Example: rectangular slits 

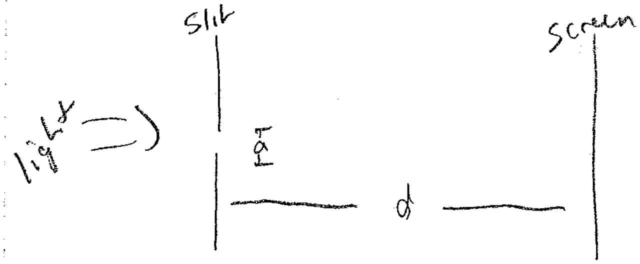
If one dimension is long then structure dominated by other dimension



compressed in y, spread out in x

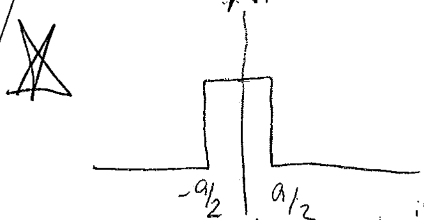
See Mathematica plots

Application: Single Slit



Aperture function

AF



$FT(\sqcap_a) = \frac{1}{\sqrt{2\pi}}\int_{-a/2}^{a/2} e^{ikx}\,dx = \frac{1}{\sqrt{2\pi}} \frac{e^{ikx}}{k}\Big|_{-a/2}^{a/2}$

$= \frac{1}{\sqrt{2\pi}}\, 2\, \frac{a}{i}\, \frac{e^{ika/2} - e^{-ika/2}}{2\,ka/2}$

$\boxed{= \frac{1}{\sqrt{2\pi}}\, a \,\mathrm{sinc}\frac{ka}{2}}$

What is diffraction pattern of a single slit?

If it's long + thin, we can neglect the Y-dimension

$$E \propto \hat{\mathcal{F}}(AF)$$

~~This AF is the "rectangular pulse" studied early, scaled to width a, so~~

$$E = E_0 \,\mathrm{sinc}\left(\frac{k_x a}{2}\right)$$

X, Y = coordinates on screen

Using defn of $k_x = \frac{kX}{d}$

$$\boxed{I(X) = I_0 \,\mathrm{sinc}^2 \frac{kaX}{2d}}$$

→ this is the pattern you see!

If it's rectangular, but not long enough to neglect the Y-dimension, you just get another sinc for Y.

$$\boxed{I(x) = I_0 \,\mathrm{sinc}^2\left(\frac{kaX}{2d}\right)\mathrm{sinc}^2\left(\frac{kbY}{2d}\right)}$$

(b is length of slit in Y-dimension)

Plots of this in PPT file ~~on next two pages~~

(for $\frac{k}{2d} = 1$, just to see shape)

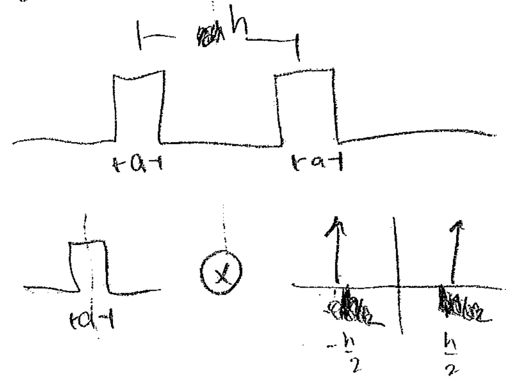
→ long + thin → ≈ delta function in Y direction

Application of the Convolution Theorem

The convolution theorem can make some problems trivial. For example:

Double Slit

1D Aperture Function (ie other dimension ≈ wide)



$$\mathcal{F}(\Omega) = \mathcal{F}(\Omega) \times \mathcal{F}(\uparrow\uparrow)$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}} a \,\text{sinc}\, \frac{k_x a}{2}}_{\text{as done previously}} \times \frac{1}{\sqrt{2\pi}} \underbrace{\int_{-\infty}^{\infty} (\uparrow\uparrow)\, e^{ik_x x}\, dx}_{\underbrace{e^{-ik_x h/2} + e^{+ik_x h/2}}_{2\cos \frac{k_x h}{2}}}$$

$k_x = \frac{kX}{d}$

$$\boxed{\mathcal{F}(\Omega) = \frac{1}{2\pi} 2a \,\text{sinc}\, \frac{k_x a}{2} \cos \frac{k_x h}{2}}$$

zeroes/maxima of this probably given in 123

~~This should look familiar to those who remember the lab!~~

times cos²

(h > a so cos oscillates faster)

$$I = I_0 \,\text{sinc}^2 \frac{kYa}{d} \cos^2 \frac{kYh}{d}$$

double slit pattern!

Piece of cake!

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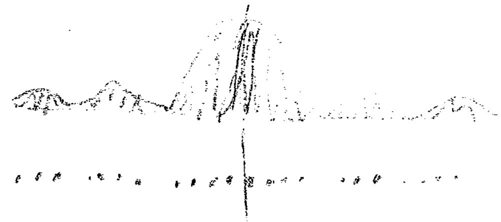
Discussion

Double Slit $I = I_0 \, \text{sinc}^2 \frac{kxa}{2d} \cos^2 \frac{kxh}{2d}$ Typically $a \ll h$

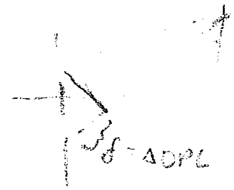


Maxima given by $\cos^2$ function

$$\frac{kxh}{2d} = m\pi$$

$$x = \frac{2d}{(2\pi/\lambda)h} m\pi = \frac{md\lambda}{h}$$

maxima $\boxed{x = \frac{md\lambda}{h}}$ same as formula from 106 book

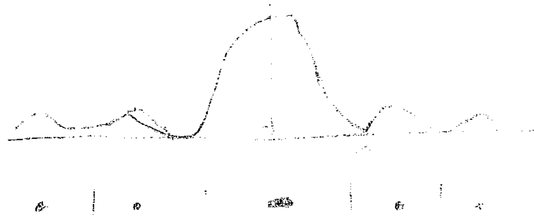


similarly

minima $x = (m+\frac{1}{2})\frac{\lambda d}{h}$

Note: overall intensity modulated by sinc function

Single Slit $I = I_0 \, \text{sinc}^2 \frac{kxa}{2d}$



minima same as sinc function
(except $x=0$ maximum)

minima: $\frac{kxa}{2d} = m\pi$

$$y = \frac{2d}{(2\pi/\lambda)a} m\pi$$

minima $\boxed{y = \frac{md\lambda}{a}}$ $m \neq 0$ same as 106 book
(handwaving argument there)

Note: overall intensity modulated by sinc function