day 6 pg 1

χ depends on freq $\quad \tilde{P}(\omega) = \epsilon_0 \tilde{\chi}(\omega) \tilde{E}(\omega)$ $\quad \tilde{P} = \tilde{P}_0 e^{i(k\cdot r - \omega t)}$, $\tilde{E} = \tilde{E}_0 e^{i(k\cdot r - \omega t)}$

complex: could have phase shift between $\tilde{P}$ and $\tilde{E}$

Back to P+N's wave eqn

$$\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial J_{free}}{\partial t} + \mu_0 \frac{\partial^2 P}{\partial t^2} - \frac{1}{\epsilon_0} \nabla(\nabla \cdot P)$$

Dielectrics $\quad J_{free} = 0$

Isotropic for now $\quad \nabla \cdot P = 0$

then $\nabla^2 E - \mu_0 \epsilon_0 \frac{\partial^2 E}{\partial t^2} = \mu_0 \frac{\partial^2 P}{\partial t^2}$ $\quad$ Plug in, divide by $e^{i(k\cdot r - \omega t)}$

$$-k^2 \tilde{E}_0 - \frac{1}{c^2}(-\omega^2)\tilde{E}_0 = \mu_0 (-\omega^2)\left(\epsilon_0 \tilde{\chi}(\omega)\right) \tilde{E}_0$$

$1/c^2$

$$\boxed{k^2 = \frac{\omega^2}{c^2}\left(1 + \tilde{\chi}(\omega)\right)}$$

$\frac{\omega}{k} = \frac{c}{\sqrt{1+\tilde{\chi}}}$

Compare $\epsilon_r = 1 + \chi$

$n = \sqrt{\epsilon_r} = \sqrt{1+\chi}$ from before

Difference $\tilde{\chi}$ could be imaginary!

define complex index of refraction

$$\boxed{\tilde{N} = \sqrt{1+\tilde{\chi}}}$$

aside: $\sqrt{\tilde{\chi}} = \sqrt{r e^{i\theta}} = \sqrt{r}\, e^{i\theta/2}$

real part of $\tilde{N}$: call "n" like before

imag. part of $\tilde{N}$: call "κ" (kappa, some books "k")

$$\boxed{\tilde{N} = n + i\kappa}$$

if $\kappa \neq 0 \Rightarrow$ absorption!

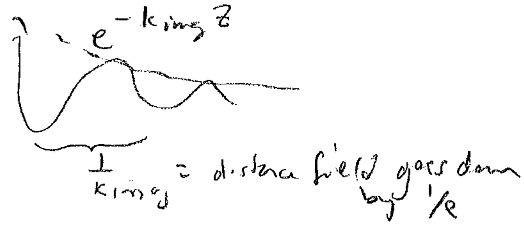
proof: $\tilde{k}$ = wave vector $= \frac{\omega}{c}\sqrt{1+\tilde{\chi}} = \frac{\omega}{c}(n + i\kappa)$ also complex! $\rightarrow$ aside: then $\tilde{E}, \tilde{B}$ not in phase!

$\tilde{E} = \tilde{E}_0 e^{i(kz - \omega t)}$ $\quad$ simple case when $k = k\hat{z}$

$= \tilde{E}_0 e^{i[(k_{real} + i k_{imag})z - \omega t]} = \tilde{E}_0 \underbrace{e^{-k_{imag} z}}_{\text{exp. decay! absorption depth}} \underbrace{e^{i(k_{real} z - \omega t)}}_{\text{controls wavelength}}$

Day 6 pg 2

k_{imag} = decay constant of field



$\alpha = 2 k_{imag}$ often used instead

$\frac{1}{\alpha} = \frac{1}{2 k_{imag}}$ called "absorption depth."

factor of 2 because energy ~ E^2
as we'll see later

Day 6 pg 3

How to predict $\tilde{N}$ theoretically?

→ Lorentz model of driven damped harmonic oscillator. {See handout!}

soln was $\tilde{x}_0 = \frac{qE_0}{m} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}$ $\tilde{x} = \tilde{x}_0 e^{-i\omega t}$

position of electron

change to $e^{i(\vec{k}\cdot\vec{r} - \omega t)}$

OK because $\cos(\vec{k}\cdot\vec{r} - \omega t)$ doesn't vary much over scale of $\tilde{x}_0$ motion

This oscillatory motion of $\vec{x}$ (electron) produces an electric dipole that changes in time

$\vec{p} = q\vec{x}$ for each electron

dipole moment

$\vec{P} = \frac{\text{total dipoles}}{\text{volume}} = q\vec{x}N$ ← # electrons/volume (some books: "n")

polarization

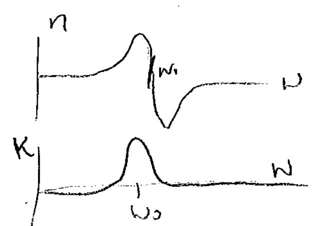
$$\tilde{P} = qN\tilde{x} = (qN)\left(\frac{qE_0}{m}\right)\frac{1}{\omega_0^2 - i\omega\gamma - \omega^2} e^{i(\vec{k}\cdot\vec{r} - \omega t)} \rightarrow \tilde{E}!$$

$\vec{P} = \epsilon_0 \chi \vec{E}$, so

$$\boxed{\tilde{\chi} = \underbrace{\frac{q^2 N}{\epsilon_0 m}}_{\omega_p^2} \frac{1}{\omega_0^2 - i\omega\gamma - \omega^2}}$$

with $\omega_p = \sqrt{\frac{q^2 N}{\epsilon_0 m}}$ = "plasma freq"

$$\boxed{\underbrace{\tilde{N}}_{n + iK} = \sqrt{1 + \tilde{\chi}}}$$ → can figure out n and K!



Final modification: if different electrons have eg. different restoring forces, they might have different spring constants → diff ω_0's.
So when you add up all dipoles to get $\vec{P}$, get fractions w/ diff ω_0

$$\boxed{\tilde{\chi} = \sum_j \frac{f_j \omega_{p_j}^2}{\omega_{0_j}^2 - i\omega\gamma_j - \omega^2}}$$

f_j = % of "j-type" of electrons
"oscillator strength"

Driven/Damped Harmonic Oscillator, by Dr. Colton
Physics 471 – Optics (driven)

"driven-damped harmonic oscillator"

"Lorentz model": charge on a spring driven back & forth,
...by oscillating electric field $E = E_0 \cos(-\omega t)$
→ I'm using $\cos(-\omega t)$ instead of $\cos(+\omega t)$ so it matches time dependence of standard traveling EM wave, $\cos(kx-\omega t)$
...with velocity-dependent damping, γ (units of γ chosen such that force = $\gamma m v$)

$$\Sigma F = m\ddot{x}$$
$$F_{driving} + F_{spring} + F_{damping} = m\ddot{x}$$
$$qE_0 \cos(-\omega t) - kx - \gamma m\dot{x} = m\ddot{x}$$
$$\ddot{x} + \gamma\dot{x} + \frac{k}{m}x = \frac{qE_0}{m}\cos(-\omega t) \qquad \rightarrow \text{Let } \omega_0 = \sqrt{\frac{k}{m}}$$
$$\boxed{\ddot{x} + \gamma\dot{x} + \omega_0^2 x = \frac{qE_0}{m}\cos(-\omega t)}$$ This is the equation of motion we need to solve

Guess $x = x_0 \cos(-\omega t + \phi)$ as solution $\rightarrow$ $\tilde{x} = x_0 e^{i\phi} e^{i(-\omega t)}$

$\tilde{x} = \tilde{x}_0 e^{i(-\omega t)}$ (ϕ is lumped in with complex $\tilde{x}_0$)

Derivatives: $\dot{\tilde{x}} = (-i\omega)\tilde{x}_0 e^{i(-\omega t)}$

$\ddot{\tilde{x}} = (-i\omega)^2 \tilde{x}_0 e^{i(-\omega t)}$

Plug into boxed equation, also changing cosine into exponential:
$$(-i\omega)^2 \tilde{x}_0 e^{i(-\omega t)} + \gamma(-i\omega)\tilde{x}_0 e^{i(-\omega t)} + \omega_0^2 \tilde{x}_0 e^{i(-\omega t)} = \frac{qE_0}{m} e^{i(-\omega t)}$$

Divide by $e^{i(-\omega t)}$, factor out $\tilde{x}_0$: $\tilde{x}_0\left(-\omega^2 + \gamma(-i\omega) + \omega_0^2\right) = \dfrac{qE_0}{m}$

$$\boxed{\tilde{x}_0 = \frac{qE_0}{m}\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma}}$$ The answer in compact form!!

Or more specifically, $\tilde{x} = \dfrac{qE_0}{m}\dfrac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} e^{-i(wt)}$

What does it mean? To understand better, we need to convert to polar form. Need first to get real & imaginary parts.

Work with the complex part: (a little like "rationalizing the denominator")
$$\frac{1}{\omega_0^2 - \omega^2 - i\omega\gamma} \times \frac{\omega_0^2 - \omega^2 + i\omega\gamma}{\omega_0^2 - \omega^2 + i\omega\gamma} = \frac{\omega_0^2 - \omega^2 + i\omega\gamma}{\left(\omega_0^2 - \omega^2\right)^2 + (\omega\gamma)^2}$$ (now in real + imaginary form)

Real part of $\tilde{x}_0 = \dfrac{qE_0}{m}\dfrac{\omega_0^{\,2} - \omega^2}{\left(\omega_0^{\,2} - \omega^2\right)^2 + (\omega\gamma)^2}$

Imaginary part $= \dfrac{qE_0}{m}\dfrac{\omega\gamma}{\left(\omega_0^{\,2} - \omega^2\right)^2 + (\omega\gamma)^2}$

Need to convert to polar form: $\sqrt{\mathrm{Re}^2 + \mathrm{Im}^2}\angle\theta$

A little algebra: $\boxed{\sqrt{\mathrm{Re}^2 + \mathrm{Im}^2} = \dfrac{qE_0}{m}\dfrac{1}{\sqrt{\left(\omega_0^{\,2} - \omega^2\right)^2 + (\omega\gamma)^2}}}$

$$\boxed{\theta = \tan^{-1}\left(\frac{\mathrm{Im}}{\mathrm{Re}}\right) = \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^{\,2} - \omega^2}\right)}$$

(Note: this angle is not correct if in II or III quadrant.)

So we have: $\tilde{x}_0 = \dfrac{qE_0}{m}\dfrac{1}{\sqrt{\left(\omega_0^{\,2} - \omega^2\right)^2 + (\omega\gamma)^2}}\exp\left[i\tan^{-1}\left(\dfrac{\omega\gamma}{\omega_0^{\,2} - \omega^2}\right)\right]$

Remembering that the magnitude of $\tilde{x}_0$ is the amplitude of oscillation and the phase of $\tilde{x}_0$ is the phase of the oscillation, our final answer is:

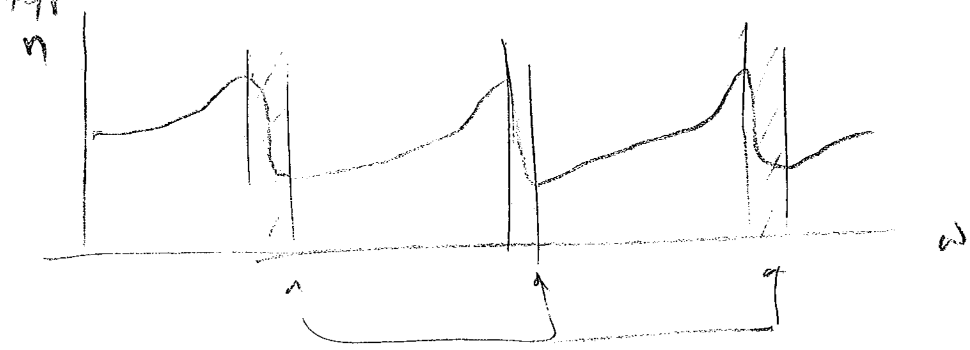
$$\boxed{x(t) = \frac{qE_0}{m}\frac{1}{\sqrt{\left(\omega_0^{\,2} - \omega^2\right)^2 + (\omega\gamma)^2}}\cos\left(-\omega t + \tan^{-1}\left(\frac{\omega\gamma}{\omega_0^{\,2} - \omega^2}\right)\right)}$$

(assuming phase angle is in I or IV quadrant)

Most of this was really just algebra… the physics was finished after the boxed equation for $\tilde{x}_0$ labeled “the answer in compact form” on the previous page.

day 6 pg 4

typical material

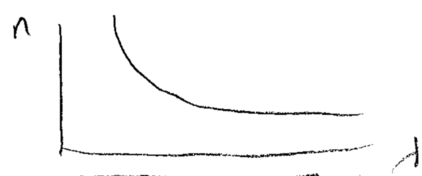


"anomalous dispersion"

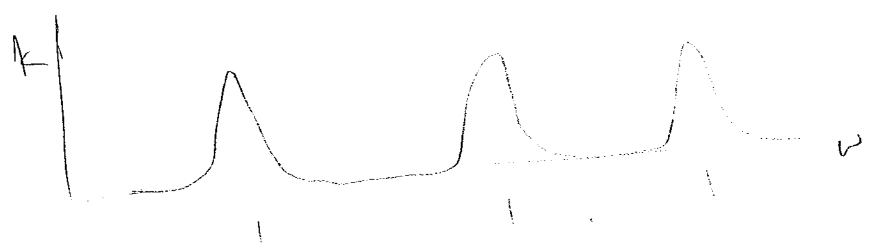
"normal dispersion" n increases w/ ω

ie. n decreases w/ λ for ω < first ω_0 resonance

for long λ



Hecht picture pg 72
fig 3-40
← in powerpoint



different resonances → diff absorptions

happens if eg different atoms cause different spring constants

end of lecture 6!