Fourier series and transforms: a summary by Dr Colton

Fourier Series

- Any reasonably well-behaved periodic function (period = T) can be written as a sum of sines and cosines, as:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nt}{T}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nt}{T}\right)$$

  The a_n and b_n numbers are called the "Fourier coefficients" and represent the amount of each cosine and sine term present in the original function.

- a_0 represents the average value of the function, and is calculated by:

$$a_0 = \frac{1}{T}\int_0^T f(t)dt$$

- The other a_n coefficients can be calculated by the formula:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos\left(\frac{2\pi nt}{T}\right)dt$$

- The b_n coefficients can be calculated by the formula:

$$b_n = \frac{2}{T}\int_0^T f(t)\sin\left(\frac{2\pi nt}{T}\right)dt$$

- All integrals can be done from $-T/2$ to $-T/2$, or for that matter over any full period, instead of from 0 to T, if it makes things easier.

Symmetry Notes:
- If the function $f(t)$ is even, only the cosine terms will be present. The b_n coefficients will all be zero.
- If the function $f(t)$ is odd, only the sine terms will be present. The a_n coefficients will all be zero.

Notation Notes:
- Often the constant term in the series expansion is written as "$a_0/2$" instead of just "a_0". That makes a_0 twice as big as in my definition. People do that so that the general a_n formula will also work for a_0.
- The equations are often written in terms of ω_0 instead of in terms of T, with $\omega_0 = 2\pi/T$. This is my personal preference. That is done so that it's apparent **all terms in the series are multiples of the lowest frequency, $\boldsymbol{\omega_0}$**, called the "fundamental frequency". The series can therefore be written like this:

$$f(t) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega_0 t) + \sum_{n=1}^{\infty} b_n \sin(n\omega_0 t)$$

  The formulas for the coefficients can be written like this:

$$a_n = \frac{2}{T}\int_0^T f(t)\cos(n\omega_0 t)\,dt$$
$$b_n = \frac{2}{T}\int_0^T f(t)\sin(n\omega_0 t)\,dt$$

- P&W write the fundamental frequency as $\Delta\omega$ instead of ω_0. That's not standard notation.

- Sometimes people write the expansion in terms of complex exponentials instead of sines and cosines, using Euler's identity $e^{ix} = \cos x + i \sin x$ to combine the a_n and b_n coefficients into a single (complex) coefficient, typically called c_n. *P&W* shows on page 14 that if you write the series as

$$f(t) = \sum_{n=-\infty}^{\infty} c_n \, e^{-in\omega_0 t}$$

then the c_n coefficients are related to the a_n and b_n coefficients as follows:

$$c_n = \begin{cases} \frac{a_{-n} - ib_{-n}}{2} & \text{for } n < 0 \\ a_0 & \text{for } n = 0 \\ \frac{a_n + ib_n}{2} & \text{for } n > 0 \end{cases}$$

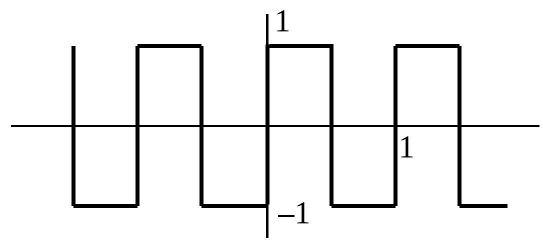
The formula to obtain the c_n coefficients is:

$$c_n = \frac{1}{T} \int_0^T f(t) e^{in\omega_0 t} dt$$

- In an exponential series like that, sometimes $f(t)$ is expanded in terms of $e^{+in\omega_0 t}$ instead of $e^{-in\omega_0 t}$. In that case, the equations for c_n would need to be modified accordingly.
- The same procedure can be done with functions of x instead of functions of t. In that case the spatial period L is used instead of the temporal period T, and the symbol k (rads/meter) is used in place of ω (rads/second).

Example 1: Square wave (infinite, repeating)

As an example, consider this function, plotted as shown:



$$f(t) = \begin{cases} -1, & \text{for } -\frac{1}{2} < t < 0 \\ 1, & \text{for } 0 < t < \frac{1}{2} \end{cases}$$

(repeated with a period of 1)

In this case, the period is 1, so the fundamental frequency is $\omega_0 = 2\pi$. All of the terms in the series will have angular frequencies that are multiples of 2π. The average value of the function is 0, so $a_0 = 0$. Additionally, the function is odd, so the expansion will contain only sine terms. A formula for the coefficients of the sine terms for this specific case can be obtained by performing the b_n integral:

$$\begin{aligned} b_n &= \frac{2}{T} \int_0^T f(t) \sin(n\omega_0 t) \, dt \\ &= \frac{2}{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(t) \sin(2\pi n t) \, dt \\ &= 2 \left(\int_{-\frac{1}{2}}^{0} (-1) \sin(2\pi n t) \, dt + \int_0^{\frac{1}{2}} (1) \sin(2\pi n t) \, dt \right) \\ &= 2 \left(\frac{\cos(2\pi n t)}{2\pi n} \bigg|_{-\frac{1}{2}}^{0} - \frac{\cos(2\pi n t)}{2\pi n} \bigg|_{0}^{\frac{1}{2}} \right) \\ &= \frac{4(1 - \cos(\pi n))}{2\pi n} \end{aligned}$$

$$= \frac{2(1-\cos(\pi n))}{\pi n}$$

That expression is equal to 0 for even values of n, and equal to $\frac{4}{\pi n}$ for odd values of n, so with the terms explicitly written out the series looks like this:

$$f(t) = \frac{4}{\pi}\sin(2\pi t) + \frac{4}{3\pi}\sin(6\pi t) + \frac{4}{5\pi}\sin(10\pi t) + \frac{4}{7\pi}\sin(14\pi t) + \cdots$$

In terms of the complex exponential representation, the series is:

$$f(t) = \cdots + \left(-\frac{2i}{5\pi}\right)e^{i10\pi t} + \left(-\frac{2i}{3\pi}\right)e^{i6\pi t} + \left(-\frac{2i}{\pi}\right)e^{i2\pi t} + \frac{2i}{\pi}e^{-i2\pi t} + \frac{2i}{3\pi}e^{-i6\pi t} + \frac{2i}{5\pi}e^{-i10\pi t} + \cdots$$

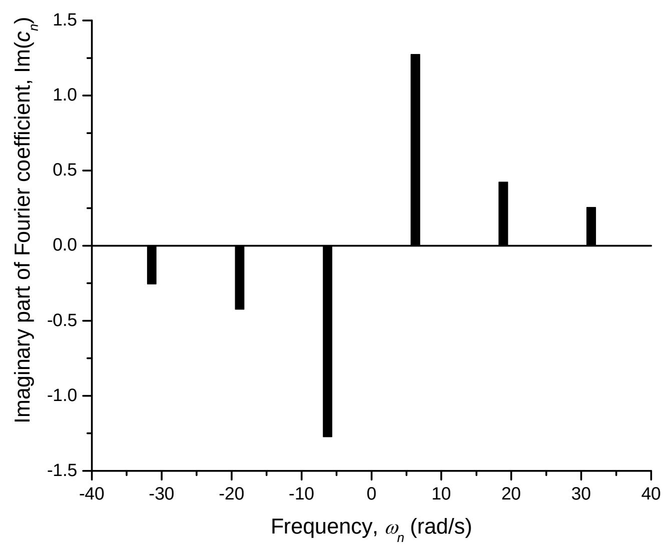
It would be a good "exercise for the reader" to verify these c_n coefficients using the integral formula above. One can also verify that this equation and the sine equation are identical using the identity $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$.

The set of Fourier coefficients can be thought of as a list like this:

$$\left\{\ldots, -\frac{2i}{5\pi}, 0, -\frac{2i}{3\pi}, 0, -\frac{2i}{\pi}, 0, \frac{2i}{\pi}, 0, \frac{2i}{3\pi}, 0, \frac{2i}{5\pi}, \ldots\right\}$$

It can also be thought of as a table of ordered pairs*, or even as a plot:

$\omega_n = n\omega_0$	c_n
…	…
-10π	$-2i/(5\pi)$
-8π	0
-6π	$-2i/(3\pi)$
-4π	0
-2π	$-2i/\pi$
0	0
2π	$2i/\pi$
4π	0
6π	$2i/(3\pi)$
8π	0
10π	$2i/(5\pi)$
…	…



* The values in the right hand column may initially seem to you to be the negative of what they should be. However these are correct, because the way the complex series has been defined in terms of $f(t) = \sum_{n=1}^{\infty} c_n\, e^{-in\omega_0 t}$ means that, for example, the $e^{i10\pi t}$ term has $\omega_n = -10\pi$, rather than $\omega_n = 10\pi$ as would perhaps be more intuitive.

Fourier Transforms

If the period of a function *increases*, then the spacing between frequency components that form the *x*-axis of the plot *decreases*. And in the limit of infinite period, where the function is not periodic at all, then an amazing thing happens: the spacing between frequencies goes to zero and instead of a set of discrete c_n points we have a continuous function! That is called a Fourier transform.

In place of ω_n we now have simply ω as the *x*-axis. Similarly, in place of $n\omega_0$ in the exponential, we have just ω. And instead of a summation over n in the series expansion equation, we have an integral over ω:

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} c(\omega) e^{-i\omega t} d\omega$$

Here the factor of $\frac{1}{\sqrt{2\pi}}$ has been arbitrarily added in to make this equation and the next one look symmetric. It has no physical significance.

Instead of the previous integrals for a_n and b_n, it can be shown via the Fourier integral theorem (*P&W* pg 16) that the proper equation for $c(\omega)$ is now:

$$c(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt$$

The function $c(\omega)$ is called the "Fourier Transform" of $f(t)$. The function $f(t)$ is called the "Inverse Fourier Transform" of $c(\omega)$.

Symmetry Notes:

- In general, $c(\omega)$ is a complex function. It's perhaps unexpected to get complex numbers from the transform of a real function. However, notice that with the way things are defined, that happens with the c_n coefficients of a Fourier series as well.
- If $f(t)$ is purely real, then $c(-\omega)$ = the complex conjugate of $c(\omega)$.
- If $f(t)$ is a purely real even function, then $c(\omega)$ = a purely real even function. That's the equivalent of an even periodic function giving rise to only cosine terms in the Fourier series.
- If $f(t)$ is a purely real odd function, then $c(\omega)$ = a purely imaginary odd function. That's the equivalent of an odd periodic function giving rise to only sine terms in the Fourier series.

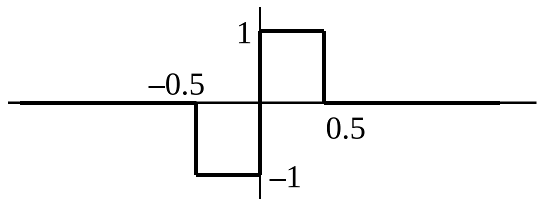
Notation Notes:

- Unfortunately "$c(\omega)$" is not the standard notation for the Fourier transform of $f(t)$. The Fourier transform of $f(t)$ is typically labeled $f(\omega)$. This is unfortunate in my opinion since the two functions both called "f", namely $f(t)$ and $f(\omega)$, are not the same function at all! Other ways of labeling the Fourier transform of $f(t)$ include $F(\omega)$, $\mathcal{F}(f(t))$, $FT\{f(t)\}$, etc.
- Factors of $\sqrt{2\pi}$ are not always included this way. Two alternate methods are:
  - If $1/\sqrt{2\pi}$ is not included in the $f(t)$ equation, the $c(\omega)$ equation will need to have $1/(2\pi)$ in it.
  - Sometimes a factor of $1/(2\pi)$ is included with the $f(t)$ equation; in that case $c(\omega)$ has no leading constant at all.
- The equations to calculate the Fourier transform and the inverse Fourier transform differ only by the sign of the exponent of the complex exponential. Many sources define the Fourier transform with $e^{i\omega t}$, in which case the $c(\omega)$ equation has $e^{-i\omega t}$ in it. Be careful.

Example 2: Square wave pulse (finite, nonrepeating)

Consider this function, plotted as shown:

$$f(t) = \begin{cases} -1, & \text{for } -\frac{1}{2} < t < 0 \\ 1, & \text{for } 0 < t < \frac{1}{2} \end{cases}$$



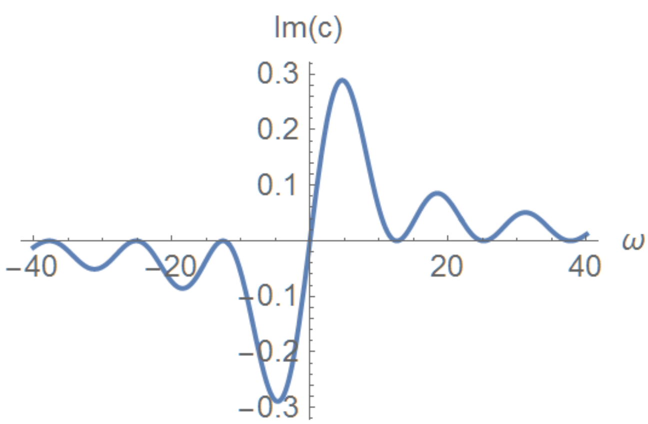
(non-repeating)

Just as the c_n coefficients of the Fourier series in Example 1 represent how much of each frequency component is present in $f(t)$, the $c(\omega)$ function in this example represents how much of each frequency component is present. The only difference is that for the periodic function in Example 1 there are only certain discrete frequency components present, whereas here there are contributions from *all* frequencies—a continuum of frequencies.

The Fourier transform function can be calculated using the $c(\omega)$ formula:

$$\begin{aligned} c(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{i\omega t} dt \\ &= \frac{1}{\sqrt{2\pi}} \left(\int_{-\frac{1}{2}}^{0} (-1) e^{i\omega t} dt + \int_{0}^{\frac{1}{2}} (1) e^{i\omega t} dt \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(-\frac{e^{i\omega t}}{i\omega}\bigg|_{-\frac{1}{2}}^{0} + \frac{e^{i\omega t}}{i\omega}\bigg|_{0}^{\frac{1}{2}} \right) \\ &= \frac{1}{\sqrt{2\pi}} \left(\frac{-1 + e^{-\frac{i\omega}{2}}}{i\omega} + \frac{e^{\frac{i\omega}{2}} - 1}{i\omega} \right) \\ &= \frac{1}{\sqrt{2\pi}} \frac{2}{i\omega} \left(\frac{e^{\frac{i\omega}{2}} + e^{-\frac{i\omega}{2}}}{2} - 1 \right) \\ &= \sqrt{\frac{2}{\pi}} \frac{i}{\omega} \left(1 - \cos\frac{\omega}{2} \right) \end{aligned}$$

That function can be plotted, and looks like this:



Many similarities between this plot and the plot of the Fourier series coefficients in Example 1 are apparent.