Bessel Functions, by Dr Colton
Physics 471, Winter 2017

The Bessel functions, $J_\alpha(x)$ are a series of functions, that:

(a) come up often, especially in partial differential equations
(b) have interesting properties
(c) are well understood and have been studied for centuries

They are typically only used for positive values of *x*. Here are the first four plotted.

Plots

BesselJ[α,x] is a built-in *Mathematica* function just like Sin[x], and gives $J_\alpha(x)$

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In[25]:= Plot[BesselJ[0, x], {x, 0, 20}]
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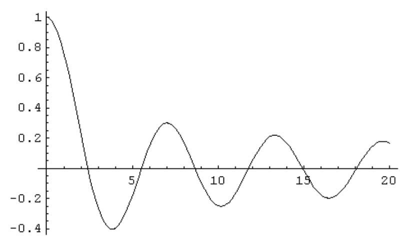


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Out[25]= - Graphics -
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$J_0(x)$
crosses zero at 2.405, 5.520, 8.654, …

← The only one that is not zero at the origin

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In[26]:= Plot[BesselJ[1, x], {x, 0, 20}]
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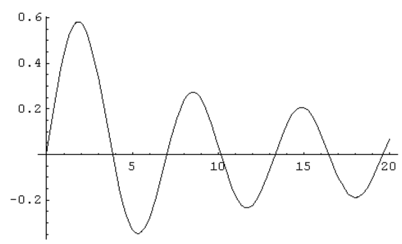


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Out[26]= - Graphics -
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$J_1(x)$
crosses zero at 3.832, 7.016, 10.173, …

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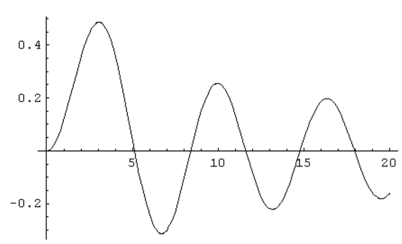
In[27]:= Plot[BesselJ[2, x], {x, 0, 20}]
```

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Out[27]= - Graphics -
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$J_2(x)$
crosses zero at 5.136, 8.417, 11.620, …

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In[28]:= Plot[BesselJ[3, x], {x, 0, 20}]
```

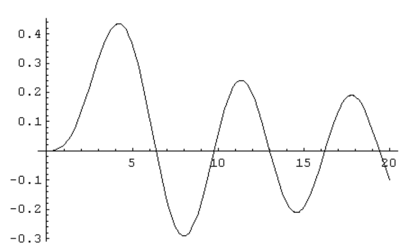


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Out[28]= - Graphics -
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$J_3(x)$
crosses zero at 6.380, 9.761, 13.015, …

Comparison with sines & cosines:

	Sines/Cosines	**Bessel functions**
1.	Two oscillatory functions: sin(x) and cos(x). Sometimes one of them is not used, due to the symmetry of the problem.	Two oscillatory functions for each α: $J_\alpha(x)$ and $Y_\alpha(x)$. Typically Y_α (also sometimes labeled N_α) is not used because it's infinite at the origin.
2.	You determine the value of sin(x) or cos(x) for arbitrary x by using a calculator or computer program.	You determine the value of $J_\alpha(x)$ for arbitrary x by using a calculator or computer program.
3.	$\sin(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+1}}{(2k+1)!}$ $\cos(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k}}{(2k)!}$	$J_\alpha(x)=\sum_{k=0}^{\infty}\frac{(-1)^k x^{2k+\alpha}}{k!(k+\alpha)!2^{2k+\alpha}}$
	<u>Consider just sin(x):</u>	<u>Consider just $J_\alpha(x)$ for one α, say $\alpha = 0$:</u> (similar things hold true for all α's)
4.	The zeroes of sin(x) are at $x = \pi, 2\pi, 3\pi$, etc. x = "$m\pi$" is the m^{th} zero	The zeroes of $J_0(x)$ are at $x \approx 2.405, 5.520, 8.654$, etc. x = "u_{0m}" is the m^{th} zero
5.	Using the zeroes in the argument, sin($m\pi x$) has m antinodes in the interval from 0 to 1. And at $x = 1$ (the boundary), sin($m\pi x$) = 0 for all m.	Using the zeroes in the argument, $J_0(u_{0m}r)$ has m antinodes in the circular region $r \le 1$. And at $r = 1$ (the boundary), $J_0(u_{0m}r) = 0$ for all m.
6.	The differential equation satisfied by $f = \sin(x)$ is $f'' + f = 0$. The differential equation satisfied by $f = \sin(m\pi x)$ is $f'' + (m\pi)^2 f = 0$.	The differential equation satisfied by $f = J_0(x)$ is $x^2f'' + xf' + (x^2 - 0^2)f = 0$. The differential equation satisfied by $f = J_0(u_{0m}x)$ is $x^2f'' + xf' + (u_{0m}^2x^2 - 0^2)f = 0$. $0^2 \to \alpha^2$ for other α's
7.	sin($n\pi x$) is orthogonal to sin($m\pi x$) on the interval (0,1): $\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = \begin{cases} 0, if\ n \ne m \\ \frac{1}{2}, if\ n = m \end{cases}$	$J_0(u_{0n}x)$ is orthogonal to $J_0(u_{0m}x)$ on the interval (0,1), with respect to a weighting function of x: $\int_0^1 xJ_0(u_{0n}x)J_0(u_{0m}x)dx = \begin{cases} 0, if\ n \ne m \\ \frac{1}{2}\left(J_1(u_{0m})\right)^2, if\ n = m \end{cases}$ Similar orthogonality holds for other values of α.

Additionally, the Bessel functions are related to sines/cosines through this integral formula:

$$J_\alpha(x) = \frac{1}{\pi}\int_0^\pi \cos(\alpha\theta - x\sin\theta)d\theta$$

Quote from Mary Boas, in *Mathematical Methods in the Physical Sciences*: "In fact, if you had first learned about sin(nx) and cos(nx) as power series solutions of $y'' = -n^2y$, instead of in elementary trigonometry, you would not feel that Bessel functions were appreciably more difficult or strange than trigonometric functions. Like sines and cosines, Bessel functions are solutions of a differential equation; they are tabulated and their graphs can be drawn; they can be represented as a series; and a large number of formulas about them are known."