Complex Numbers Summary, by Dr Colton
Physics 471 – Optics

We will be using complex numbers as a tool for describing electromagnetic waves. *P&W* has a short section in Chapter 0 on the fundamentals of complex numbers, section 0.2, but here is my own summary.

Colton's short complex number summary:

- A complex number $x + iy$ can be written in rectangular or polar form, just like coordinates in the x-y plane.
  - The rectangular form is most useful for adding/subtracting complex numbers.
  - The polar form is most useful for multiplying/dividing complex numbers.
- The polar form $(A,\ \theta)$ can be expressed as a complex exponential $Ae^{i\theta}$.
- For example, consider the complex number $3 + 4i$:
  - = (3, 4) in rectangular form,
  - = (5, 53.13º) in polar form, and
  - = $5e^{i53.13^\circ}$ or $5e^{0.9273i}$ in complex exponential form, since 53.13º = 0.9273 rad.
- The complex exponential form follows directly from Euler's equation: $e^{i\theta} = \cos\theta + i\sin\theta$, and by looking at the x- and y-components of the polar coordinates.
- By the rules of exponents, when you multiply/divide two complex numbers in polar form, $(A_1,\ \theta_1)$ and $(A_2,\ \theta_2)$, you get:
  - multiply: $A_1e^{i\theta 1} \times A_2e^{i\theta 2} = A_1A_2e^{i(\theta 1+\theta 2)} = (A_1A_2,\ \theta_1+\theta_2)$
  - divide: $A_1e^{i\theta 1} \div A_2e^{i\theta 2} = (A_1/A_2)e^{i(\theta 1-\theta 2)} = (A_1/A_2,\ \theta_1-\theta_2)$
- I like to write the polar form using this notation: $A\angle\theta$. The "$\angle$" symbol is read as, "at an angle of". Thus you can write:

  $(3 + 4i) \times (5 + 12i)$
  $= 5\angle 53.13^\circ \times 13\angle 67.38^\circ$
  $= 65\angle 120.51^\circ$ (since $65 = 5 \times 13$ and $120.51^\circ = 53.13^\circ + 67.38^\circ$)

Representing waves as complex numbers:

Suppose you have an electromagnetic wave traveling in the z-direction and oscillating in the y-direction. The equation for the wave would be this:

$$\mathbf{E} = E_0\hat{\mathbf{y}}\cos(kz - \omega t + \phi)$$

It's often helpful to represent that type of function with complex numbers, like this:

$$\mathbf{E} = E_0\hat{\mathbf{y}}\cos(kz - \omega t + \phi) \rightarrow \mathbf{E} = E_0\hat{\mathbf{y}}\boldsymbol{e}^{i(kz-\omega t+\phi)}$$

It's understood that this is just a temporary mathematical substitution. If you want to know the **real oscillation**, you take the **real part** of the complex exponential, i.e. turn it back into a cosine.

$$\rightarrow \mathbf{E} = E_0e^{i\phi}\hat{\mathbf{y}}\boldsymbol{e}^{i(kz-\omega t)}$$
$$\rightarrow \mathbf{E} = \tilde{E}_0\hat{\mathbf{y}}\boldsymbol{e}^{i(kz-\omega t)}$$

Now $\tilde{E}_0$ is actually a complex number whose magnitude is E_0, the wave's amplitude, and whose phase is ϕ, the phase of the oscillating cosine wave. This type of trick will make the math much easier for some calculations we need to do.