**Derivation of Poynting's Theorem**
**Dr Colton, Winter 2019**

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \text{(dot with } \mathbf{B}\text{)} \rightarrow \mathbf{B} \cdot (\nabla \times \mathbf{E}) = -\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t} \qquad \text{(dot with } \mathbf{E}\text{)} \rightarrow \mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mu_0 \mathbf{E} \cdot \mathbf{J} + \epsilon_0 \mu_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$$

$$\mathbf{B} \cdot (\nabla \times \mathbf{E}) = -\frac{1}{2} \frac{\partial (B^2)}{\partial t}$$

$$\mathbf{E} \cdot (\nabla \times \mathbf{B}) = \mu_0 \mathbf{E} \cdot \mathbf{J} + \frac{\epsilon_0 \mu_0}{2} \frac{\partial (E^2)}{\partial t}$$

Bottom equation minus top equation…

$$\underbrace{\mathbf{E} \cdot (\nabla \times \mathbf{B}) - \mathbf{B} \cdot (\nabla \times \mathbf{E})}_{-\nabla \cdot (\mathbf{E} \times \mathbf{B})} = \mu_0 \mathbf{E} \cdot \mathbf{J} + \underbrace{\frac{\epsilon_0 \mu_0}{2} \frac{\partial (E^2)}{\partial t} + \frac{1}{2} \frac{\partial (B^2)}{\partial t}}_{\mu_0 \frac{\partial}{\partial t}\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)}$$

Multiply both sides by $-1/\mu_0$…

$$\underbrace{\nabla \cdot \left(\frac{1}{\mu_0} (\mathbf{E} \times \mathbf{B})\right)} = \underbrace{-\mathbf{E} \cdot \mathbf{J}} - \frac{\partial}{\partial t} \underbrace{\left(\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}\right)}$$

This is the energy per unit time, per area, transported by the fields; define it with the symbol **S** called the **Poynting vector**.

This is the rate at which work is done by the fields on the medium, per volume; P&W call it $\frac{\partial u_{medium}}{\partial t}$.

This is the energy stored in the **E** and **B** fields, per volume (see Appendices 2C and 2D); P&W call it $u_{field}$.

This then is Poynting's Theorem; written in differential form with the terms slightly rearranged it is:

$$\boxed{-\frac{\partial u_{field}}{\partial t} = \frac{\partial u_{medium}}{\partial t} + \nabla \cdot \mathbf{S}}$$

And in integral form where $U_{field}$ represents the total energy stored in the fields in some volume of space and $W$ is the work done by the fields on the charges in that volume, it looks like this:

$$\boxed{-\frac{\partial U_{field}}{\partial t} = \frac{\partial W}{\partial t} + \oint \mathbf{S} \cdot d\mathbf{A}}$$

(I used the divergence theorem to change the volume integral of $\nabla \cdot \mathbf{S}$ into the integral of $\mathbf{S} \cdot d\mathbf{A}$ over the surface bounding the volume.)

It is a statement of conservation of energy. If energy which has been stored in the fields is lost ($\partial U_{field}/\partial t$ is negative, which makes the LHS positive), then that energy either goes into doing work on the charges in that volume of space, or else it gets transported out of the volume. The rightmost term can be thought of as an energy flux (per time).

In a region of space where there is no medium, it becomes:

$$-\frac{\partial u_{field}}{\partial t} = \nabla \cdot \mathbf{S}$$

Compare that statement of energy conservation to the equation of continuity, which is a statement of charge conservation; you can see they have the identical mathematical form:

$$-\frac{\partial \rho}{\partial t} = \nabla \cdot \mathbf{J}$$