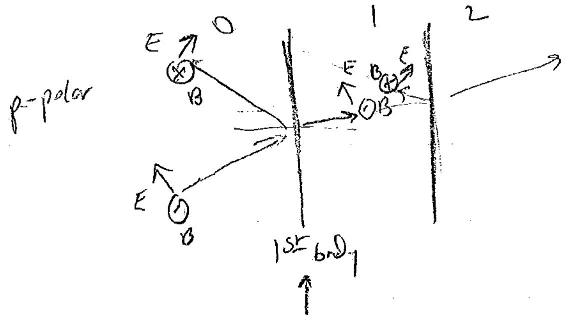
Solving 2-body via matrices



(1) $E_{\parallel} = E_{\parallel}$
(2) $B_{\parallel} = B_{\parallel}$ $\quad |E| \frac{n}{c} = |E| \frac{n}{c}$

(1) $\underline{(E_{0\rightarrow} + E_{0\leftarrow})\cos\theta_0 = (E_{1\rightarrow} + E_{1\leftarrow})\cos\theta_1}$

(2) $B_{0\rightarrow} - B_{0\leftarrow} = B_{1\rightarrow} - B_{1\leftarrow}$

$\underline{n_0(E_{0\rightarrow} - E_{0\leftarrow}) = n_1(E_{1\rightarrow} - E_{1\leftarrow})}$

(1) and (2) summarized $\boxed{\begin{pmatrix} \cos\theta_0 & \cos\theta_0 \\ n_0 & -n_0 \end{pmatrix}\begin{pmatrix} E_{0\rightarrow} \\ E_{0\leftarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_1 & \cos\theta_1 \\ n_1 & -n_1 \end{pmatrix}\begin{pmatrix} E_{1\rightarrow} \\ E_{1\leftarrow} \end{pmatrix}}$

0 | 1 | 2

2nd body

remember phase shifts!

(1) $\left(E_{1\rightarrow} e^{ik_1 l_1 \cos\theta_1} + E_{1\leftarrow} e^{-ik_1 l_1 \cos\theta_1}\right)\cos\theta_1 = E_{2\rightarrow}\cos\theta_2$

(2) $n_1\left(E_{1\rightarrow} e^{ik_1 l_1 \cos\theta_1} - E_{1\leftarrow} e^{-ik_1 l_1 \cos\theta_1}\right) = n_2 E_{2\rightarrow}$

$$\boxed{\begin{pmatrix} e^{ik_1 l_1 \cos\theta_1}\cos\theta_1 & e^{-ik_1 l_1 \cos\theta_1}\cos\theta_1 \\ n_1 e^{i(\ )} & -n_1 e^{-i(\ )} \end{pmatrix}\begin{pmatrix} E_{1\rightarrow} \\ E_{1\leftarrow} \end{pmatrix} = \begin{pmatrix} \cos\theta_2 & 0 \\ n_2 & 0 \end{pmatrix}\begin{pmatrix} E_{2\rightarrow} \\ 0 \end{pmatrix}}$$

let $\boxed{\beta_1 = k_1 l_1 \cos\theta_1}$

$$\begin{pmatrix} E_{0\rightarrow} \\ E_{0\leftarrow} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta_0 & \cos\theta_0 \\ n_0 & -n_0 \end{pmatrix}^{-1} \underbrace{\begin{pmatrix} \cos\theta_1 & \cos\theta_1 \\ n_1 & n_1 \end{pmatrix}\begin{pmatrix} \cos\theta_1 e^{i\beta_1} & \cos\theta_1 e^{-i\beta_1} \\ -n_1 e^{i\beta_1} & -n_1 e^{i\beta_1} \end{pmatrix}^{-1}}_{\text{only depends on material 1, call it } M_1} \begin{pmatrix} \cos\theta_2 & 0 \\ n_2 & 0 \end{pmatrix}}_{A}\begin{pmatrix} E_{2\rightarrow} \\ 0 \end{pmatrix}$$

Strategy: compute $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

$$\begin{pmatrix} E_{0\rightarrow} \\ E_{0\leftarrow} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} E_{2\rightarrow} \\ 0 \end{pmatrix}$$

Divide by $E_{0\rightarrow}$

$$\begin{pmatrix} 1 \\ r \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} t_{02} \\ 0 \end{pmatrix}$$

$$1 = a_{11} t_{02} \longrightarrow \boxed{t_{02} = \frac{1}{a_{11}}}$$

$$r = a_{21} t_{02} \longrightarrow \boxed{r = \frac{a_{21}}{a_{11}}}$$

really does give the same as eqns from early in chapter (4.11, 4.12)
(Trust me...)

Why? 1) Easy for computer to do matrix calculations

2) Easy to extend to additional layers.

Each layer adds two new matrices of form M_1.
For N internal layers...

$$\boxed{A = \begin{pmatrix} \cos\theta_0 & \cos\theta_0 \\ n_0 & -n_0 \end{pmatrix}^{-1} \left(\prod_{j=1}^{N} M_j \right) \begin{pmatrix} \cos\theta_{N+1} & 0 \\ n_{N+1} & 0 \end{pmatrix}}$$

$$\boxed{M_j = \begin{pmatrix} \cos\theta_j & \cos\theta_j \\ n_j & n_j \end{pmatrix} \begin{pmatrix} \cos\theta_j e^{i\beta_j} & \cos\theta_j e^{-i\beta_j} \\ n_j e^{i\beta_j} & -n_j e^{-i\beta_j} \end{pmatrix}^{-1}}$$

with θ_j from Snell's Law $\boxed{n_0 \sin\theta_0 = n_j \sin\theta_j}$

and $\boxed{\beta_j = k_j l_j \cos\theta_j}$

$\boxed{k_j = \frac{2\pi}{\lambda_j} = \frac{2\pi n_j}{\lambda_{vac}}}$

Simplifying M_j's: use $\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \frac{1}{\det M} \begin{pmatrix} D & -B \\ -C & A \end{pmatrix}$

$$M_j = \begin{pmatrix} \cos\theta_j & \cos\theta_j \\ n_j & -n_j \end{pmatrix} \frac{1}{-n_j\cos\theta_j - n_j\cos\theta_j} \begin{pmatrix} -n_j e^{-i\beta_j} & -\cos\theta_j e^{-i\beta_j} \\ -n_j e^{i\beta_j} & \cos\theta_j e^{i\beta_j} \end{pmatrix}$$

$$= \frac{1}{2n_j\cos\theta_j} \underbrace{\begin{pmatrix} \cos\theta_j & \cos\theta_j \\ n_j & -n_j \end{pmatrix} \begin{pmatrix} n_j e^{i\beta_j} & \cos\theta_j e^{-i\beta_j} \\ n_j e^{i\beta_j} & -\cos\theta_j e^{i\beta_j} \end{pmatrix}}$$

$$= \frac{1}{2n_j\cos\theta_j} \begin{pmatrix} n_j\cos\theta_j\left(e^{i\beta_j} + e^{-i\beta_j}\right) & \\ & \text{etc.} \end{pmatrix}$$

$$\boxed{M_j = \begin{pmatrix} \cos\beta_j & -\frac{i\sin\beta_j\cos\theta_j}{n_j} \\ -\frac{in_j\sin\beta_j}{\cos\theta_j} & \cos\beta_j \end{pmatrix}}$$

Simplifying A: use same matrix inversion

$$A = \frac{1}{-n_0\cos\theta_0 - n_0\cos\theta_0} \begin{pmatrix} -n_0 & -\cos\theta_0 \\ -n_0 & \cos\theta_0 \end{pmatrix} \left[\prod M_j\right] \begin{pmatrix} \cos\theta_{N+1} & 0 \\ n_{N+1} & 0 \end{pmatrix}$$

$$\boxed{A = \frac{1}{2n_0\cos\theta_0} \begin{pmatrix} n_0 & \cos\theta_0 \\ n_0 & -\cos\theta_0 \end{pmatrix} \left[\prod M_j\right] \begin{pmatrix} \cos\theta_{N+1} & 0 \\ n_{N+1} & 0 \end{pmatrix}}$$

S-polarization: HW problem

$$M_j = \begin{pmatrix} \cos\beta_j & -\frac{i\sin\beta_j}{n_j\cos\theta_j} \\ -in_j\cos\theta_j\sin\beta_j & \cos\beta_j \end{pmatrix}$$

and $$A = \frac{1}{2n_0\cos\theta_0} \begin{pmatrix} n_0\cos\theta_0 & 1 \\ n_0\cos\theta_0 & -1 \end{pmatrix} \left[\prod M_j\right] \begin{pmatrix} 1 & 0 \\ n_{N+1}\cos\theta_{N+1} & 0 \end{pmatrix}$$

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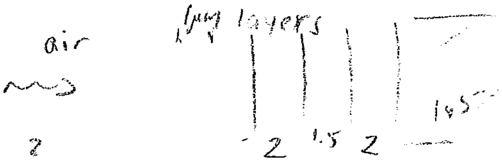
Example: normal incidence $\theta = 0°$



$R = ?$

$\beta_j = k_j l_j \cos\theta_j$

$= \frac{2\pi}{\lambda/n}(1)$ in microns

$\beta_1 = \frac{2\pi}{\lambda} 2$, $\beta_2 = \frac{2\pi}{\lambda} 1.5$, etc.

$$A = \frac{1}{2 n_0 \cos\theta_0} \begin{pmatrix} n_0 & \cos\theta_0 \\ n_0 & -\cos\theta_0 \end{pmatrix} \left[\prod_{j=1}^{N} \begin{pmatrix} \cos\beta_j & \frac{-i \sin\beta_j \cos\theta_j}{n_j} \\ \frac{-i n_j \sin\beta_j}{\cos\theta_j} & \cos\beta_j \end{pmatrix} \right] \begin{pmatrix} \cos\theta_{N+1} & 0 \\ n_{N+1} & 0 \end{pmatrix}$$

$$M_j = \begin{pmatrix} \cos\beta_j & -\frac{i}{n_j}\sin\beta_j \\ -i n_j \sin\beta_j & \cos\beta_j \end{pmatrix}$$ since $\theta_j = 0°$

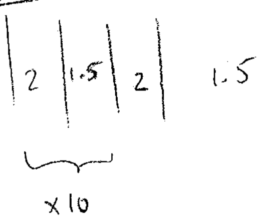
$$= \frac{1}{2(1)(1)} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} M_1 M_2 M_3 \begin{pmatrix} 1 & 0 \\ 1.5 & 0 \end{pmatrix}$$

Matrix Multi. done with Mathematica, $r = a_{21}/a_{11}$, $R = |r|^2$ plotted

handout w/plot

Note: use "." for Matrix Multiplication

Different structure



Again, done w/ Mathem. $A = \frac{1}{2(1)(1)} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} (M_1 M_2)^{10} M_3 \begin{pmatrix} 1 & 0 \\ 1.5 & 0 \end{pmatrix}$

plot on handout

Note: Use MatrixPower command for $(M_1 M_2)^{10}$

not $(M_1 M_2)$^10

Or use P+W's tips in section 4.8 for raising matrices to a power

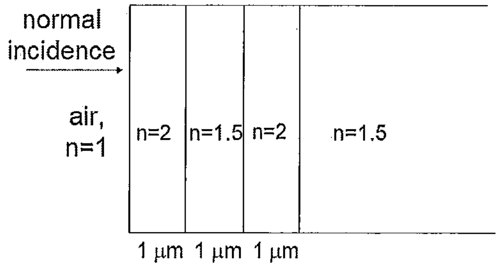
using Sylvester's theorem

$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^2 = \ldots$

I'm sure that would have made second plot go much faster!

Quarter Wave Stack — Quiz

Multilayers – Example 1



multilayer-1.nb

In[1]:=
```
beta[lambda_, n_] = 2 Pi n / lambda
```

Out[1]= $\dfrac{2\,n\,\pi}{\text{lambda}}$

In[2]:=
```
M[lambda_, n_] =
  {{Cos[beta[lambda, n]], -I Sin[beta[lambda, n]] / n},
   {-I n Sin[beta[lambda, n]], Cos[beta[lambda, n]]}}
```

Out[2]= $\left\{\left\{\text{Cos}\left[\dfrac{2\,n\,\pi}{\text{lambda}}\right],\ -\dfrac{i\,\text{Sin}\left[\frac{2\,n\,\pi}{\text{lambda}}\right]}{n}\right\},\ \left\{-i\,n\,\text{Sin}\left[\dfrac{2\,n\,\pi}{\text{lambda}}\right],\ \text{Cos}\left[\dfrac{2\,n\,\pi}{\text{lambda}}\right]\right\}\right\}$

In[3]:=
```
MatrixForm[%]
```

Out[3]//MatrixForm=

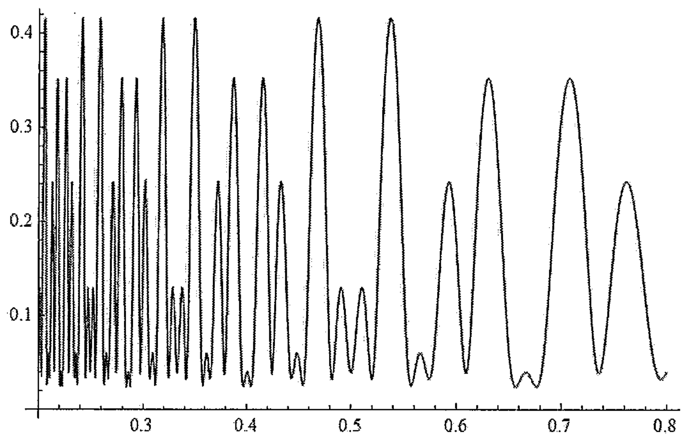
$$\begin{pmatrix} \text{Cos}\left[\frac{2\,n\,\pi}{\text{lambda}}\right] & -\frac{i\,\text{Sin}\left[\frac{2\,n\,\pi}{\text{lambda}}\right]}{n} \\ -i\,n\,\text{Sin}\left[\frac{2\,n\,\pi}{\text{lambda}}\right] & \text{Cos}\left[\frac{2\,n\,\pi}{\text{lambda}}\right] \end{pmatrix}$$

In[4]:=
```
A[lambda_] =
  1/2 {{1, 1}, {1, -1}} . M[lambda, 2] . M[lambda, 1.5] .
    M[lambda, 2] . {{1, 0}, {1.5, 0}};
```

In[5]:=
```
r[lambda_] = A[lambda][[2, 1]] / A[lambda][[1, 1]];
```

In[6]:=
```
Plot[Abs[r[lambda]^2], {lambda, 0.2, 0.8}]
```

Out[6]=



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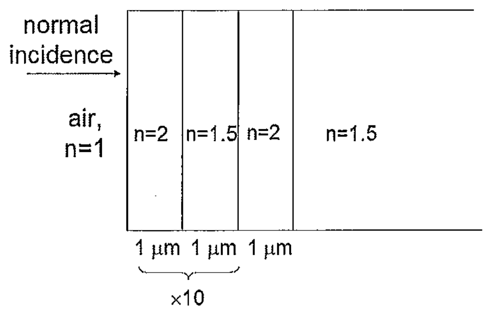
Multilayers – Example 2



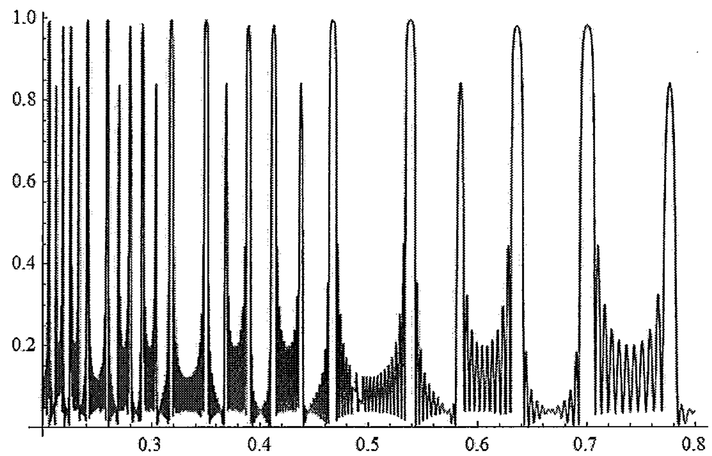
multilayer-1.nb

```
In[7]:= Anew[lambda_] =
          1/2 {{1, 1}, {1, -1}} .
            MatrixPower[M[lambda, 2] . M[lambda, 1.5] , 10].
            M[lambda, 2] . {{1, 0}, {1.5, 0}};

In[8]:= rnew[lambda_] = Anew[lambda][[2, 1]] / Anew[lambda][[1, 1]];

In[9]:= Plot[Abs[rnew[lambda]^2], {lambda, 0.2, 0.8}]
```

Out[9]=



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