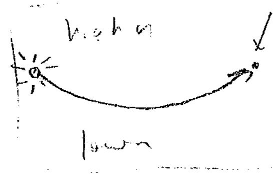
Chap 9. Light as rays

Suppose n changes with position.



which emitted ray ends up here and what is its path!

light path curved

→ wave fronts not plane waves!

Try to solve wave eqn

$$\nabla^2 E = \frac{1}{v^2}\frac{\partial^2 E}{\partial t^2}$$

$$\nabla^2 E - \frac{n^2(\vec{r})}{c^2}\frac{\partial^2 E}{\partial t^2} = 0$$

Trial soln: instead of $E = \vec{E_0}\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ ← function of r

$$\underline{E = \vec{E_0}(\vec{r})\, e^{i(k_{vac} R(r) - \omega t)}}$$

ie if n not varying, $R = \frac{\vec{k}\cdot\vec{r}}{k_{vac}}$

(= z in vacuum w/ plane waves in z dir.)

$\frac{\partial}{\partial t} \rightarrow (-i\omega)$

$\frac{\partial^2}{\partial t^2} \rightarrow -\omega^2$

then we have $\frac{\omega^2}{c^2} = k_{vac}^2$ in right hand term

divide by this

$$\frac{1}{k_{vac}^2}\nabla^2\left(\vec{E_0}(\vec{r})\, e^{ik_{vac}R}\, \cancel{e^{-i\omega t}}\right) + n^2(\vec{r})\, \vec{E_0}(\vec{r})\, e^{ik_{vac}R}\, \cancel{e^{-i\omega t}} = 0$$

Worked problem: what is $\nabla^2\left(\vec{E_0}(\vec{r})\, e^{ik_{vac}R(\vec{r})}\right)$?

use defn $\nabla^2 f = \nabla\cdot(\nabla f)$ for each component

for E_{0x}: $= \nabla\cdot\left[E_{0x}(\vec{r})\,\nabla(e^{ik_{vac}R}) + e^{ik_{vac}R}\,\nabla(E_{0x}(\vec{r}))\right]$ using product rule

use more product rules, get factors of $\nabla(R)$ from chain rule

combine E_x, E_y, E_z results

plug back into wave eqn

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$$\left[\vec{\nabla} R \cdot \vec{\nabla} R - n^2\right] \vec{E}_0(\vec{r}) = \frac{\nabla^2 \vec{E}_0(\vec{r})}{k_{vac}^2} + \frac{i}{k_{vac}} \nabla^2 R + \frac{2i}{k_{vac}} \hat{x} (\nabla E_{0x} \cdot \nabla R) + \hat{y} + \hat{z} \text{ terms}$$

$\boxed{\text{Big Approx:}}$ short wavelength $\lambda_{vac} \to 0$, $\frac{1}{k_{vac}} \to 0$ } only LHS survives

Rays: can only describe features that are large compared to a wavelength

$$\vec{\nabla} R(\vec{r}) \cdot \vec{\nabla} R(\vec{r}) = n^2(\vec{r})$$

$$(\nabla R)^2 = n^2$$

magn. of $\nabla R = n$

$$\boxed{\nabla R(\vec{r}) = n(\vec{r})\, \hat{s}} \quad \text{Eikonal Eqn}$$

$\hookrightarrow \hat{s}$ can change with $\vec{r}$ also!

HW 9.2 Poynting vector in ∇R direction $\Rightarrow \hat{s}$ = dir of energy flow
(not assigned) at each pt in space

Actual light path is $/\!/$ to $\hat{s}$ at every spot.

Remember Fermat's principle? (path = least time)

proof:

$$\underbrace{\nabla \times (\nabla R)}_{\substack{=0 \\ \text{curl of gradient}}} = \nabla \times (n \hat{s})$$

Integrate over surface:

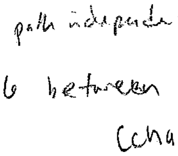
$$\underbrace{\int_S \nabla \times (n\hat{s})\, da}_{} = 0$$

$$\oint_P (n\hat{s}) \cdot d\vec{\ell} = 0 \quad \text{(Stoke's Thm)}$$

Recall scalar potential function
$\oint \vec{E} \cdot d\vec{l} = 0$
$\to V = -\int \vec{E} \cdot d\vec{l}$

Corollary: $\int_A^B (n\hat{s}) \cdot d\vec{\ell}$ = path independent

Keep in mind both n and $\hat{s}$ are changing w/ position



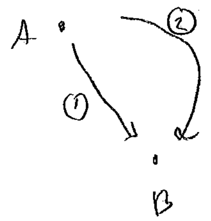
pick one path from B to A
then all other return paths have to give (the integral) the same thing!
So that $\oint = 0$

$$\int_A^B n \cos\theta\, d\ell = \text{path independent}$$

$\cos\theta$ = angle between $\hat{s}$ and $d\vec{\ell}$
(changing during integral)

$\int_A^B n \cos\theta \, dl$ = path independent

↳ θ = angle between $\vec{s}$ and $d\vec{l}$, potentially changing during integral



Consider these two paths

1. Actual light path, $\cos\theta = 1$ always during integral
2. other light path, $\cos\theta \neq 1$

Integral is the same

$$\left.\int_A^B n \, dl\right|_{\text{path 1}} = \left.\int_A^B n \cos\theta \, dl\right|_{\text{path 2}}$$

always less than 1: if you take it out, result will be bigger

$$\left.\int_A^B n \, dl\right|_{\text{path 1}} < \left.\int_A^B n \, dl\right|_{\text{path 2}}$$

$$\boxed{OPL = \int_A^B n \, dl}$$ = smallest possible value when the path = actual light path

"optical path length" used before as OPL = n·d

I.e. figure out which of all possible paths has the minimal OPL and that's the correct one!

Fermat: "figure out ... minimal time ...!"

reason: $v = \frac{x}{t} \rightarrow t = \frac{x}{v} = \int \frac{dl}{v}$ if v is changing → $v = c/n$

$$t = \frac{1}{c}\underbrace{\int n \, dl}_{OPL}$$

minimal OPL = minimal time.

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How does this compare to $\int_A^B n\,d\ell$? (it's always $\leq$, since $\cos\theta$ always ≤ 1)

(= only when path entirely in direction of energy flow)

$\boxed{\int_A^B n\,d\ell = \text{OPL}}$

(should look familiar – before $n \cdot \ell = \text{OPL}$)

→ Figure out which of all possible paths has the minimal OPL, then $\int_A^B n \cos\theta\,d\ell$ must still be $\leq$ to this (=, obviously)

$\int_A^B (n\hat{s}) \cdot d\vec{\ell}\,\Big|_{\text{actual path}}$ = one with smallest possible OPL

= $c \times$ smallest possible time!

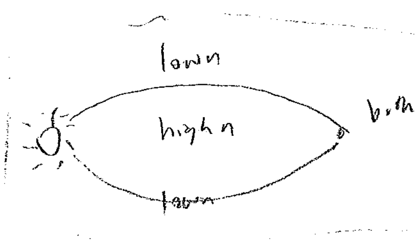
since $t = \frac{\text{distance}}{\text{velocity}}$

time for path $= \int \frac{d\ell}{c/n(\vec{r})}$ if n changes

$= \frac{1}{c} \times \text{OPL}$

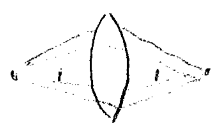
Note: doesn't apply to crystals where n depends on direction, not just position

What if more than one path has same OPL?



both rays get here!

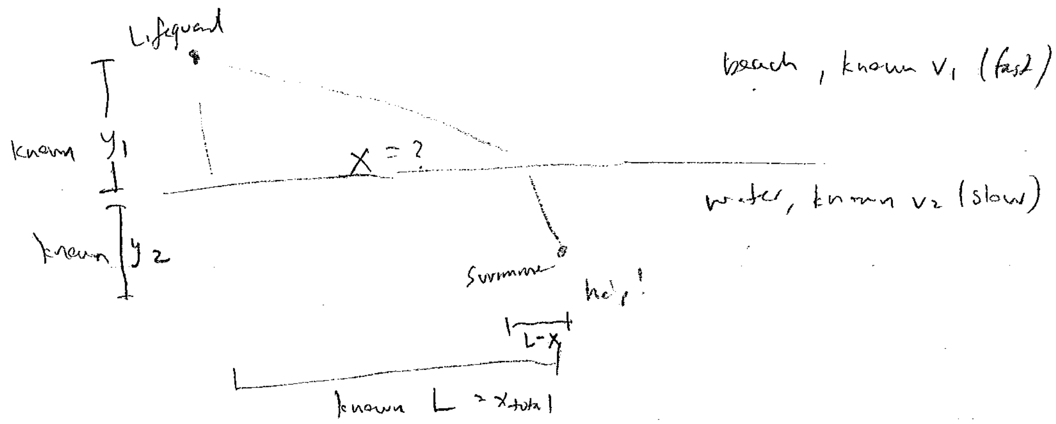
Common example: a lens!



an infinite # of paths have same OPL

Fermat's principle of least time → Snell's Law

"Lifeguard Problem" (given to 125 class)



$v = \frac{x}{t}$

$$\text{time} = t_1 + t_2$$

$$= \frac{d_1}{v_1} + \frac{d_2}{v_2}$$

$$t = \frac{\sqrt{x^2 + y_1^2}}{v_1} + \frac{\sqrt{(L-x)^2 + y_2^2}}{v_2}$$

minimize: $\frac{dt}{dx} = 0 \rightarrow \frac{1}{v_1} \cdot +\frac{1}{2}(x^2 + y_1^2)^{-\frac{1}{2}}(2x) + \frac{1}{v_2} \cdot +\frac{1}{2}\left((L-x)^2 + y_2^2\right)^{-1/2} 2(L-x)(-1) = 0$

$$\frac{1}{v_1} \frac{x}{\sqrt{x^2 + y_1^2}} = \frac{1}{v_2} \frac{L-x}{\sqrt{(L-x)^2 + y_2^2}}$$

A pain to solve for x .... but look! Define θ = angle from normal

$$\frac{1}{v_1} \sin\theta_1 = \frac{1}{v_2} \sin\theta_2$$

Light → $v \sim \frac{1}{n}$

$$\boxed{n_1 \sin\theta_1 = n_2 \sin\theta_2}$$ Snell's Law

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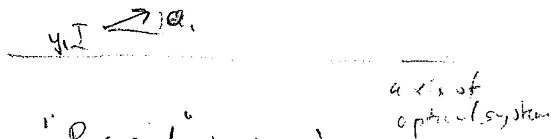
ABCD Matrices

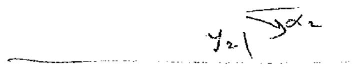
- characterize light ray at a particular spot with two parameters

$$\mathbb{X}_1 = \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$



"Paraxial": y and α are small (close to axis)

at another point, that ray will be different direction / position



$$\mathbb{X}_2 = \begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix}$$

Represent change via a 2×2 matrix!

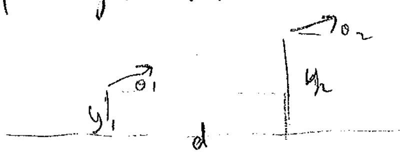
$$\mathbb{X}_2 = M \mathbb{X}_1$$

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = 4 \text{ components of } M$$

$$\det M = AD - BC$$

"Beauty": Effects of several situations → just multiply matrices together! Need to find matrices

Situation 1: straight line motion, distance d



$$\theta_2 = \theta_1$$

$$y_2 = y_1 + d\tan\theta \simeq y_1 + d\theta$$

$$\begin{pmatrix} y_2 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ \theta_1 \end{pmatrix}$$

$$M = \begin{pmatrix} 1 & d \\ 0 & 1 \end{pmatrix}$$