Day 32 pg 1

Ch 10: Diffraction

Back to wave eqn: $\nabla^2 \vec{E} = \frac{1}{v^2}\frac{\partial^2 \vec{E}}{\partial t^2}$

harmonic time dependence, $\frac{\partial}{\partial t} \to -i\omega$

$\nabla^2 \vec{E} = -k^2 \vec{E}$ (use $\frac{\omega}{v} = k$)

"vector Helmholtz eqn"

rect: $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

spherical? $\vec{E} = \vec{E}(r,\theta,\phi)$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \vec{E}}{\partial r}\right) + \underbrace{\frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial \vec{E}}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 \vec{E}}{\partial \phi^2}}_{=0}$$

simplest case: $E = E(r)$ only $\to$

If not vector, $f = \frac{A}{r}\cos(kr - \omega t)$ works. I.e. $\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial f}{\partial r}\right) = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$ ✓

$\nabla^2 E = -k^2 E$ "scalar Helmholtz eqn"

If vector that must satisfy Maxwell

guess, since it's $\perp$ to travel

$\vec{E} = \frac{A}{r}\cos(kr - \omega t)\,\hat{\phi}$ doesn't work!! HW P10.2 (assigned)

Can satisfy 1st three, but not Ampere's

$\vec{E}(r,\theta) = \frac{A\sin\theta}{r}\left(\cos(kr-\omega t) - \frac{1}{kr}\sin(kr-\omega t)\right)\hat{\phi}$ does work!

It's a pain to prove it ... in Griffith's HW, but not P10.2

But... let's say r = big

$\theta \simeq 0$ true for travel close to z-direction since E oscillation will then be in xy plane

then that becomes $\vec{E} = \frac{A}{r}\cos(kr - \omega t)\,\hat{\phi}$

Discussion of scalar vs vector in "Significance of Scalar wave approx" on pg 261

Bottom line: For r = big and $\theta \simeq \pi/2$

then if E satisfies scalar Helmholtz, $\vec{E}$ will satisfy (mostly) vector Helmholtz

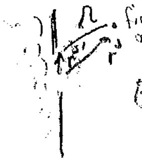
"scalar approximation"

Complex: $E = \frac{E_0 e^{ikr}}{r}$ or $= E_0 \frac{r_0}{r} e^{ikr}$ to give E_0 units of field.

Where are we going?

Huygens: light as propagation of infinite number of spherical wavelets

[Fig 10.1]



$$E(\vec{r}) \sim \int_{\text{aperture}} E(\text{at aperture})\,\frac{e^{ik\mathcal{R}}}{\mathcal{R}}\,da'$$

z

"Guess"

~~Kirchhoff~~

$\vec{\mathcal{R}} = \vec{r} - \vec{r}'$

$\mathcal{R} = \sqrt{(x-x')^2 + (y-y')^2 + z^2}$

book: $\mathcal{R} = R$

Can we derive this from the Wave Eqn? (or Helmholtz eqn)

Travel in z-direction

Expect $E = \frac{E_0}{r} e^{ikz}$

Let's make this more general,

Call it $\tilde{E}$. • Units of field
• expect something like a $\frac{1}{z}$ or $\frac{1}{r}$ dependence

Scalar Helmholtz:

$$\nabla^2 E = -k^2 E$$

Plug in guess

$$\nabla^2 \left(\tilde{E} e^{ikz}\right) = -k^2 \left(\tilde{E} e^{ikz}\right)$$

Some derivatives...

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\tilde{E} e^{ikz}\right) = -k^2 \tilde{E} e^{ikz}$$

(product rule)

$$e^{ikz}\frac{\partial^2 \tilde{E}}{\partial x^2} + e^{ikz}\frac{\partial^2 \tilde{E}}{\partial y^2} + \frac{\partial}{\partial z}\left(\tilde{E}\, ik e^{ikz} + e^{ikz}\frac{\partial \tilde{E}}{\partial z}\right) = -k^2 \tilde{E} e^{ikz}$$

Another two product rules

$$ik\left[\tilde{E}\, ik e^{ikz} + e^{ikz}\frac{\partial \tilde{E}}{\partial z}\right] + \left[e^{ikz}\frac{\partial^2 \tilde{E}}{\partial z^2} + ik e^{ikz}\frac{\partial \tilde{E}}{\partial z}\right]$$

$$\left(-k^2 \tilde{E} + 2ik\frac{\partial \tilde{E}}{\partial z} + \frac{\partial^2 \tilde{E}}{\partial z^2}\right)e^{ikz}$$

All e^{ikz}'s cancel

$-k^2 E$ term cancels w/ RHS

Result: $$\frac{\partial^2 \tilde{E}}{\partial x^2} + \frac{\partial^2 \tilde{E}}{\partial y^2} + 2ik\frac{\partial \tilde{E}}{\partial z} + \underbrace{\frac{\partial^2 \tilde{E}}{\partial z^2}}_{\approx 0} = 0$$

"paraxial wave approximation"
Amplitude of wave varies slowly in z direction so it looks much like a plane wave

The (or maybe "a") soln to this eqn: HW P10.5 (unassigned)

$$\tilde{E} = -\frac{i}{\lambda z} \iint_{\text{aperture}} E(x', y')\, e^{i\frac{k}{2z}\left[(x-x')^2 + (y-y')^2\right]} dx' dy'$$

aperture ↓ where field ≠ 0

$E(x',y')$: field strength across aperture (if varying) ← "aperture function"

do integral, then $E = \tilde{E}\, e^{ikz}$

gives answer just like our original guess with $r = \left[(x-x')^2 + (y-y')^2 + z^2\right]^{1/2}$

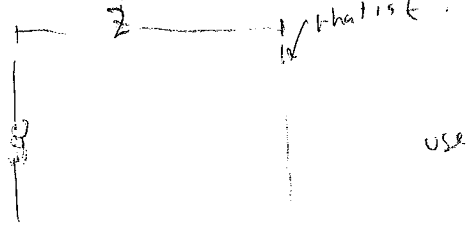
$$\simeq z\left[1 + \left(\frac{x-x'}{z}\right)^2 + \left(\frac{y-y'}{z}\right)^2\right]^{1/2}$$

$$\simeq z\left(1 + \frac{1}{2}\frac{(x-x')^2}{z^2} + \frac{1}{2}\frac{(y-y')^2}{z^2} + \dots\right)$$

but with proper $-\frac{i}{\lambda}$ out in front

just • keep zeroth order for denom. $(\frac{1}{z})$
• keep first order for exponent (need that since small change in r → big change in e^{ikr})

To actually do integral



use $(x-x')^2 = x^2 - 2xx' + x'^2$

x^2: can be taken out of integral

(same for y)

$$\boxed{E = -\frac{i}{\lambda z}\, e^{ikz}\, e^{\frac{ik}{2z}(x^2+y^2)} \iint_{\text{aperture}} E(x',y')\, e^{\frac{ik}{2z}(x'^2+y'^2)}\, e^{-i\frac{k}{z}(xx'+yy')}\, dx'dy'}$$ ☆

"The Fresnel Approximation" Eqn. 10.13

~~Babinet! if aperture = ▣ (square lets light through, circle is blocked, etc.)~~

~~then $\iint_{\square}(\ )da' - \iint_{\circ}(\ )da'$~~

~~Not so interesting?~~

~~diffraction from hair $\iint_{\text{hair}} = \iint_{\text{all space}} - \iint_{\text{slit}}$ → hair diffraction very similar to slit!~~

~~from ● (blocked circle) → diffraction like ○ (open circle) = (we'll see later) bright spot in middle! "Poisson's spot"~~

Note: There are 4 complicated integral formulas in this chapter
I only care about 2 of them really (except problem P10.6)

☆ • "Fresnel Approx." Eqn 10.13, that we just discussed

• "Fresnel's Diffraction Formula" Eqn 10.1

$$E(x,y,z) = \frac{-i}{\lambda} \iint_{aperture} E(x',y',0) \frac{e^{ik\sqrt{(x-x')^2+(y-y')^2+z^2}}}{\sqrt{\text{same}}} dx'dy'$$

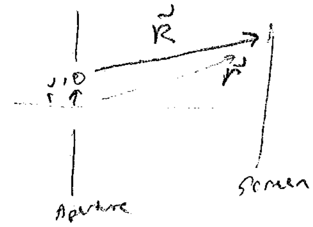
Developed before Maxwell's eqns. Not correct.
"Guessed"

• "Fresnel-Kirchhoff Diffraction formula" Eqn 10.10

same thing, w/ additional term in integral: $\frac{1+\cos\theta}{2}$

$\theta = \angle$ between $\vec{R}$ and z-axis

$\vec{R} = \vec{r} - \vec{r}'$



~ Correct, but hard to use.

~ Turns into Fresnel Approx, Eqn 10.13 w/ approximations we made earlier.

☆ • "Fraunhofer Approx" Eqn 10.19
which we'll talk about soon.

Notes on HW P10.6
→ see PPT slide

Babinet:

If aperture =  light through square, blocked by circle

then $\iint_{\square} \_\_ \, da' - \iint_{\circ} \_\_ \, da'$

Not so interesting?

Diffraction from hair:

$$E = \iint_{\text{hair}} = \underbrace{\iint_{\text{all space}}}_{\text{no features}} - \iint_{\text{slit}}$$

$|E_{\text{from hair}}|^2$ similar to $|E_{\text{single slit}}|^2$

I from ● blocked circle like I ○ open circle

solved later, Bessel function^2

bright spot in middle

bright spot in middle?

judge: Simeon Poisson (1818) "No, can't be!"
same as Poisson distribution

François Arago measured. "Yes!"

Fresnel wins award!