cylindrical coords, only $f(r)$

$$F(k_r, k_\theta) = \iint f(r)\, e^{i(k_x x' + k_y y')}\, dx'dy'$$

possible factor of $\frac{1}{2\pi}$

poll: (r, θ, ϕ)? $p+w$: (ρ, ϕ)

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$k_x = k_r \cos k_\theta \qquad k_y = k_r \sin k_\theta \qquad k_r = \sqrt{k_x^2 + k_y^2}$$

and $dx'dy' = r\,dr\,d\theta$ (book uses ρ instead of r) (Griffiths uses s)

$$F(k_r, k_\theta) = \frac{1}{2\pi}\iint f(r)\, e^{i[(k_r\cos k_\theta)(r\cos\theta) + (k_r \sin k_\theta\, r\sin\theta)]} \times r\,dr\,d\theta$$

$$= \frac{1}{2\pi}\iint f(r)\, e^{ik_r r[\cos k_\theta \cos\theta + \sin k_\theta \sin\theta)}\, r\,dr\,d\theta$$

$$= \int_0^\infty f(r)\, r \left[\frac{1}{2\pi}\int_0^{2\pi} e^{ik_r r[\cos(\theta - k_\theta)]}\, d\theta\right] dr$$

The integral in the brackets belongs to a class of functions called the Bessel functions, defined by

$$J_\alpha(x) = \frac{1}{\pi}\int_0^\pi \cos(\alpha\theta - x\sin\theta)\, d\theta$$

last eqn on Bessel function handout

Specifically, we have an $\alpha = 0$ Bessel function, since

$$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\, d\theta$$

Proof: $J_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos(x\sin\theta)\,d\theta$ since $\cos(x\sin\theta)$ = even in θ

proof $\cos(x\sin(-\theta)) = \cos(-x\sin\theta)$

$= \cos(x\sin\theta)$ ✓

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\cos\left(x\cos(\theta-\tfrac{\pi}{2})\right)d\theta$$ since $\sin\theta = \cos(\theta - \frac{\pi}{2})$

$\theta' = \theta - \pi/2$ subst., then rename back to θ

$$= \frac{1}{2\pi}\int_{-\frac{3\pi}{2}}^{\pi/2}\cos(x\cos\theta)\,d\theta$$

$$\cancel{= \frac{1}{2\pi}\int_0^{2\pi}\cos(x\cos\theta)\,d\theta}$$ ~~because integrating over full period both ways~~

$$= \frac{1}{2\pi}\int_{-\frac{3\pi}{2}}^{\frac{\pi}{2}}\left[\cos(x\cos\theta) + i\sin(x\cos\theta)\right]d\theta$$

since the sine part integrates to 0

proof: $-\frac{3\pi}{2}$ to $-\frac{\pi}{2}$ part cancels w/ $-\frac{\pi}{2}$ to $\frac{\pi}{2}$

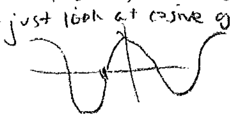
since it's "odd" with respect to $-\frac{\pi}{2}$

I.e. $\sin\left(x\cos\left(-\frac{\pi}{2}-\theta\right)\right) = -\sin\left(x\cos\left(-\frac{\pi}{2}+\theta\right)\right)$

obviously true (?) since

$\cos(-\frac{\pi}{2}-\theta) = -\cos(-\frac{\pi}{2}+\theta)$

just look at cosine graphs



$$= \frac{1}{2\pi}\int_{-3\pi/2}^{\pi/2} e^{ix\cos\theta}\,d\theta$$

$$= \frac{1}{2\pi}\int_{-k_\theta}^{2\pi-k_\theta} e^{ix\cos\theta}\,d\theta$$ because still integrating over a full period

do $\theta' = \theta + k_\theta$, then rename back to θ

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ix\cos(\theta-k_\theta)}\,d\theta$$ for any k_θ!

I.e. $F(k_r, k_\theta) = \int_0^\infty r f(r)\, J_0(k_r r)\, dr.$

↑ FT of radial function f

Bessel Function Handout

After going through handout

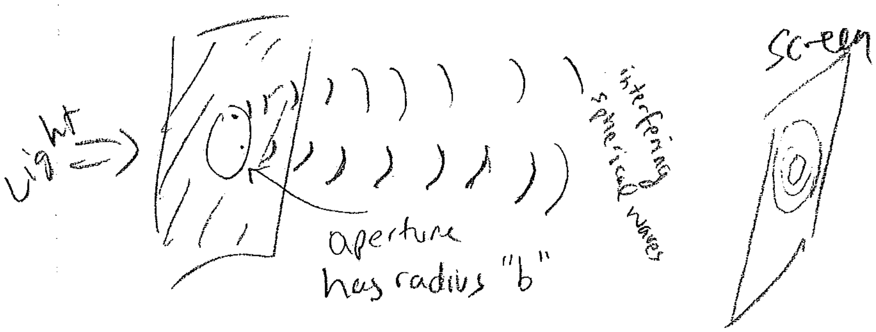
→ Some more on orthogonality
(PPT pictures)

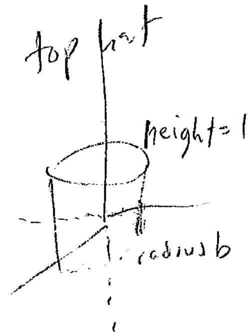
What $\int_0^1 x\, J_0(u_{0n} x)\, J_0(u_{0m} x)\, dx$ means

Now back to ~~[illegible]~~ diffraction ...

$$FT(f(r,\theta)) = \int_0^{\infty} r f(r) J_0(k_r r)$$

Diffraction from a circular aperture





The aperture function is exactly the "top hat function".

$$\mathcal{F}(\text{aperture}) = \int_0^\infty \left(\substack{\text{aperture}\\ \text{function}}\right) r J_0(r k_r)\, dr$$

F.T. for polar coordinates

$$= \int_0^b r J_0(r k_r)\, dr$$

let $r' = r k_r$

$$= \int_0^{b k_r} \left(\frac{r'}{k_r}\right) J_0(r') \left(\frac{dr'}{k_r}\right)$$

$$= \frac{1}{k_r^2} \underbrace{\int_0^{k_r b} r' J_0(r')\, dr'}_{(k_r b) J_1(k_r b)}$$

$$= \frac{b}{k_r} J_1(k_r b)$$

$$\boxed{\mathcal{F}\left(\substack{\text{top hat of}\\ \text{circle radius } b}\right) = b^2 \frac{J_1(k_r b)}{k_r b}}$$

$\frac{J_1(x)}{x}$ = "jinc" sometimes

or

P+W $\rightarrow$ $\frac{2J_1(x)}{x}$ = "jinc" so that $\mathrm{jinc}(0) = 1$ (like sinc) day 35 pg 5

$k_r = \sqrt{k_x^2 + k_y^2} = \frac{2\pi}{\lambda d}\sqrt{x^2+y^2}$

Recall that the diffraction pattern varies as the F.T., squared, with $k_r = \frac{kR}{d}$

(R = radial distance on the screen, d = distance the screen is away from the aperture, $k = \frac{2\pi}{\lambda}$)

So the diffraction pattern (shape, at least) seen on the screen is:

$$\boxed{I = I_0 \left[\frac{2 J_1\left(\frac{kRb}{d}\right)}{\left(\frac{kRb}{d}\right)} \right]^2}$$

factor of 2 so that $I_0 = I(R=0)$ since $\frac{J_1(x)}{x} = \frac{1}{2}$ at $x \to 0$

This is the "Airy ring pattern", ~~which you likely looked at in optics.~~

J_1's plotted in PPT file ~~on the next page~~.

Fraunhofer

To be technical (book's answer)

$$E = \frac{-ie^{i(\ldots)}}{\lambda z} (2\pi \, FT)$$

(book "z" instead of "d")

$$I = \frac{I_0}{\lambda^2 z^2} 4\pi^2 \left| \frac{b^2 J_1\left(\frac{kRb}{z}\right)}{\frac{kRb}{z}} \right|^2$$

$$I = \frac{I_0 (\pi 2b)^2}{\lambda^2 z^2} J_1(\ldots)$$

book's answer. But constant factors seem silly ... What is I_0? (really)

Aperture shifted from origin → very much like array theorem

$$\boxed{\text{F.T. of shifted aperture} = e^{ik_x x_o} e^{ik_y y_o} \times \text{F.T. of regular aperture}}$$

} already discussed

~~So shifting the origin just introduces phase factors. Not terribly important for this example, but can be if you have different phase factors from multiple apertures.~~

Example Problem Diffraction pattern of circle square combo, given...

circle : radius a

square : side a

square + circle each centered a distance "2a" from the origin.

We can immediately use the previous results and write down the F.T.

$$\text{F.T.} = (\text{F.T.})_{\substack{\text{shifted}\\\text{square}}} + (\text{F.T.})_{\substack{\text{shifted}\\\text{circle}}}$$

$$F.T. = \underbrace{e^{ik_x 2a}}_{\text{shift of square}} \Big[\underbrace{a \,\mathrm{sinc}\left(\tfrac{k_x a}{2}\right) \Big] \Big[a \,\mathrm{sinc}\left(\tfrac{k_y a}{2}\right)}_{\text{FT of square}} \Big]$$

$$+ \underbrace{e^{ik_x(-2a)}}_{\text{shift of circle}} \Big[\underbrace{2\pi a^2 \frac{J_1(k_r a)}{k_r a}}_{\text{FT of circle}} \Big]$$

$$F.T. = e^{-ik_x(2a)} a^2 \left[e^{i4k_x a} \,\mathrm{sinc}\frac{k_x a}{2}\,\mathrm{sinc}\frac{k_y a}{2} + 2\pi \frac{J_1(k_r a)}{k_r a} \right]$$

$$\text{So } I = I_0 \left| e^{i4\frac{kXa}{d}} \,\mathrm{sinc}\frac{kXa}{2d}\,\mathrm{sinc}\frac{kYa}{2d} + 2\pi \frac{J_1\left(\frac{k\sqrt{X^2+Y^2}}{d} a\right)}{\frac{k\sqrt{X^2+Y^2}}{d} a} \right|^2$$

complicated, but doable!

Plotted w/ Mathematica → see PPT

~~See plot on next page~~