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Complex Numbers

→ handout!

$\boxed{\text{Euler } e^{i\theta} = \cos\theta + i\sin\theta}$

$Ae^{i\theta} = A\cos\theta + iA\sin\theta$

Relations the other way

so $\boxed{A\cos\theta = \text{Real}\left[Ae^{i\theta}\right]}$

Sinusoidal function varying in time

$5\cos(\omega t + 30°) = \text{Re}\left[5e^{i(\omega t + 30°)}\right]$

$= 5e^{i30°}e^{i\omega t}$

$= 5\angle 30°$

"phasor notation"

• real part assumed

When all terms in an equation have same eg time dependence, we don't have to worry about them (ie they all cancel out)

$= 5\angle 30°$

• $e^{i\omega t}$ assumed, can do if all terms in eqn will have it

This class: $e^{i\omega t}$ assumed (really then tho?)

later

Since traveling waves will be like $\cos(kx - \omega t)$

How to add $7\angle 30° + 5\angle 100°$? (answer in polar form)

$7e^{i\pi/6} + 5e^{i5\pi/9}$

Complex Numbers Summary, by Dr Colton
Physics 471 – Optics

We will be using complex numbers as a tool for describing electromagnetic waves. *P&W* has a short section in Chapter 0 on the fundamentals of complex numbers, section 0.2, but here is my own summary.

Colton's short complex number summary:

- A complex number $x + iy$ can be written in rectangular or polar form, just like coordinates in the x-y plane.
  - The rectangular form is most useful for adding/subtracting complex numbers.
  - The polar form is most useful for multiplying/dividing complex numbers.
- The polar form (A, θ) can be expressed as a complex exponential $Ae^{i\theta}$.
- For example, consider the complex number $3 + 4i$:
  - $= (3, 4)$ in rectangular form,
  - $= (5, 53.13°)$ in polar form, and
  - $= 5e^{i53.13°}$ or $5e^{0.9273i}$ in complex exponential form, since $53.13° = 0.9273$ rad.
- The complex exponential form follows directly from Euler's equation: $e^{i\theta} = \cos\theta + i\sin\theta$, and by looking at the x- and y-components of the polar coordinates.
- By the rules of exponents, when you multiply/divide two complex numbers in polar form, (A_1, θ_1) and (A_2, θ_2), you get:
  - multiply: $A_1e^{i\theta 1} \times A_2e^{i\theta 2} = A_1A_2e^{i(\theta 1+\theta 2)} = (A_1A_2,\ \theta_1+\theta_2)$
  - divide: $A_1e^{i\theta 1} \div A_2e^{i\theta 2} = (A_1/A_2)e^{i(\theta 1-\theta 2)} = (A_1/A_2,\ \theta_1-\theta_2)$
- I like to write the polar form using this notation: $A\angle\theta$. The "$\angle$" symbol is read as, "at an angle of". Thus you can write:

  $(3 + 4i) \times (5 + 12i)$
  $= 5\angle 53.13° \times 13\angle 67.38°$
  $= 65\angle 120.51°$ (since $65 = 5 \times 13$ and $120.51° = 53.13° + 67.38°$)

Representing waves as complex numbers:

Suppose you have an electromagnetic wave traveling in the z-direction and oscillating in the y-direction. The equation for the wave would be this:

$$\vec{E} = E_0\hat{y}\cos(kz - \omega t + \phi)$$

It's often helpful to represent that type of function with complex numbers, like this:

$$\vec{E} = E_0\hat{y}\cos(kz - \omega t + \phi) \quad \rightarrow \quad \vec{E} = E_0\hat{y}e^{i(kz-\omega t+\phi)}$$

It's understood that this is just a temporary mathematical substitution. If you want to know the **real oscillation**, you take the **real part** of the complex exponential, i.e. turn it back into a cosine.

$$\rightarrow \quad \vec{E} = E_0e^{i\phi}\hat{y}e^{i(kz-\omega t)}$$

$$\rightarrow \quad \vec{E} = \tilde{E}_0\hat{y}e^{i(kz-\omega t)}$$

Now $\tilde{E}_0$ is actually a complex number whose magnitude is E_0, the wave's amplitude, and whose phase is ϕ, the phase of the oscillating cosine wave. This type of trick will make the math much easier for some calculations we need to do.

Application Numbers made up by class. Like HW 0.20

$$\_\_\_\cos(3t + \_\_\_) + \_\_\_\cos(3t + \_\_\_)$$

Select random students

$$= A\cos(3t + \phi)$$

What are A and ϕ?

Need to know $\sin x = \cos(x - 90°)$

Taught in my 123 section ....

Like adding vectors!

$$Re\left[A_1 e^{i(3t+\phi_1)} + A_2 e^{i(3t+\phi_2)} \right]$$

$$Re\left[e^{i3t} \underbrace{(A_1 \angle\phi_1 + A_2 \angle\phi_2)} \right]$$

add polar coords = add vectors

$$A \angle\phi$$

$$Re\left[e^{i3t} A e^{i\phi} \right]$$

$$Re\left[A e^{i(3t+\phi)} \right]$$

$$A\cos(3t+\phi)$$

Test with Mathematica

Challenge! Work out this answer some other way!

Bonus points for first person

Often an understood $Re[\ ]$, understood e^{i3t}, or even $e^{i(kx-\omega t)}$

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Bottom of Complex # summary

"Complex amplitude" → ~~...~~

ex: $\vec{E} = 8\hat{y}\cos(kz - \omega t + 30°)$

$\vec{E}_0$ = amplitude = $8\hat{y}$

complex amplitude = $8\angle 30°$ $\hat{y}$

= ~~...~~

incorporates phase into E_0.

Notation: regular amplitude → regular vector symbol $\vec{E}_0$

complex amplitude → new vector symbol $\tilde{E}_0$

(Griffiths notation; but P+W don't use ~)

"$\vec{E}_0$" → understood complex phase

polar coord. angle ⟷ complex phase.

$\vec{E}$ + $\vec{B}$ waves, a few quick notes

- $\vec{B}$ also has a wave equation, identical when in free space (as mentioned)
- If you plug $\vec{E} = \vec{E}_0 \cos(kx - \omega t + \phi)$ into Faraday's law...

$\tilde{E}_0 e^{i(kx - \omega t)}$ $\qquad \nabla \times \vec{E} = -\frac{\partial B}{\partial t}$

$$\vec{B} = \tilde{B}_0 e^{i(kx - \omega t)}$$

$$i(k \times \tilde{E}_0)\, \cancel{e^{i(kx - \omega t)}} = -\tilde{B}_0 (-i\omega)\, \cancel{e^{i(kx - \omega t)}}$$

$$\tilde{B}_0 = \frac{\tilde{k} \times \tilde{E}_0}{\tilde{\omega}}$$

(a) same phase!

(b) $\vec{B}_0 = \frac{1}{\omega/k} E_0 \, \hat{u}$

↳ unit vector ⊥ to $\tilde{E}_0$

$\hat{k} \times \hat{E} = \vec{B}$

$\hat{E} \times \hat{B} = \hat{k}$ direction of travel

(c) $|B_0| = \frac{1}{c} |E_0|$

small!

TOPS. 35500

Don't need to write all on board

Recall $\nabla \cdot \vec{D} = \rho_{free}$

$= 0$ in regular dielectric

$\boxed{\vec{D} = \epsilon_0 \vec{E} + \vec{P}}$ defn!

Before we used $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$ to obtain

wave eqn with $v = \frac{c}{\sqrt{\epsilon_r}}$ (if $\mu_r = 1$, ie. nonmagnetic)

ie. $n = \sqrt{\epsilon_r}$.

But does $\vec{D}$ always equal $\epsilon_0 \epsilon_r \vec{E}$? Not necessarily

$\vec{P}$ more fundamental than $\vec{D}$

(and can be complicated $\vec{E} \xrightarrow{\text{causes}} \vec{P}$, $\vec{P} \xrightarrow{\text{affects}} \vec{E}$)

Susceptibility: $\vec{P} = \epsilon_0 \chi \vec{E}$

aka χ_e, because can have a similar mag. χ

how much $\vec{P}$ produced, depends on material

convert units

compare $D = \epsilon_0 \epsilon_r E$

→ In regular materials $\epsilon_r = 1 + \chi$

This is all I need

(often $\vec{P} \propto \vec{E}$, but not always)

if linear $\boxed{\epsilon_r = 1 + \chi}$

Notes

(1) χ could be matrix: $\begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix} = \epsilon_0 \begin{pmatrix} & & \\ & \chi & \\ & & \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}$

must have this if "anisotropic"

ie. $\vec{P}$ in different direction than $\vec{E}$

(2) χ could induce nonlinearities

$$P = \epsilon_0 (\chi_1 E + \chi_2 E^2 + \dots)$$

must have this if "nonlinear"

(3) χ can depend on frequency of light

(4) χ could be complex - phase shift between $\vec{E}$ + $\vec{P}$

We'll worry about (3)+(4) now, (1) later (Ch. 5) and (2) not at all

in grad class, eg Phy 571