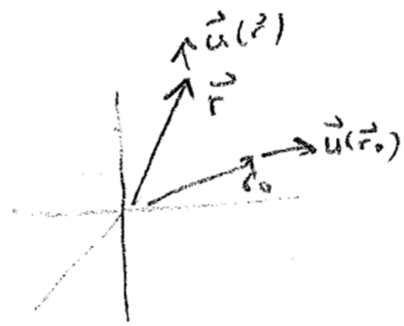
Elastic Strains

by Dr. Colton, Physics 581 (last updated: Fall 2020)

I found Kittel pg. 73-75 to be nearly incomprehensible. After doing some research and talking to an acoustics professor, I found a much better description in Fetter and Walecka, *Theoretical Mechanics of Particles and Continua*, pg 460-463. This mainly follows their treatment.



$\vec{u} =$ displacement field, describes where each point goes under a deformation.

i.e. $\vec{r}' = \vec{r} + \vec{u}(\vec{r})$ is the new position of $\vec{r}$

$\vec{r}_0' = \vec{r}_0 + \vec{u}(\vec{r}_0)$ is the new position of $\vec{r}_0$

We will consider only small displacements; do a Taylor's series, first order.

Taylor's Series Review:

In 1-D

- $f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} (x - x_0)$

For 3-D scalar functions

- $f(x,y,z) = f(x_0,y_0,z_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{y_0} (y - y_0) + \left.\frac{\partial f}{\partial z}\right|_{z_0} (z - z_0)$ =>
- $f(\mathbf{r}) = f(\mathbf{r}_0) + \nabla f|_{\mathbf{r}_0} \cdot (\mathbf{r} - \mathbf{r_0})$ =>
- $f(\mathbf{r}) - f(\mathbf{r}_0) = \sum_{j=1}^{3} \left.\frac{\partial f}{\partial x_j}\right|_{\vec{r}_0} \left(r_j - r_{0j}\right)$ where j =1, 2, 3 refer to the x,y,z components of a vector, respectively, and the ∂x_j derivatives refer to $\partial x, \partial y, \partial z$.

For 3-D vector functions: each component of $\mathbf{u} = (u_x, u_y, u_z)$ is a scalar function.
For the i[th] component:

- $u_i(\mathbf{r}) - u_i(\mathbf{r}_0) = \sum_{j=1}^{3} \left.\frac{\partial u_i}{\partial x_j}\right|_{\mathbf{r}_0} \left(r_j - r_{0j}\right)$ which is just like matrix multiplication:

$$\begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \frac{\partial u_x}{\partial z} \\ \frac{\partial u_y}{\partial x} & \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} \\ \frac{\partial u_z}{\partial x} & \frac{\partial u_z}{\partial y} & \frac{\partial u_z}{\partial z} \end{pmatrix} \begin{pmatrix} r_x - r_{0x} \\ r_y - r_{0y} \\ r_z - r_{0z} \end{pmatrix}$$

The 3×3 matrix $\frac{\partial u_i}{\partial xj}$ is called the "deformation gradient".

Break for some notation and connections to Kittel

- $\mathbf{u}$ is the same as Kittel's $\mathbf{R}$, Eq. 3.27
- The components (u_x, u_y, u_z) are the same as Kittel's (u, v, w)
- The nine $\frac{\partial u_i}{\partial xj}$ terms are Kittel's ϵ_{ij} components

Now write the deformation gradient as a sum of symmetric and anti-symmetric matrices,

$$\frac{\partial u_i}{\partial xj} = \frac{1}{2}\begin{pmatrix} \frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} & \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} + \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial \mathrm{x}} + \frac{\partial u_x}{\partial z} & \frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} + \frac{\partial u_z}{\partial z} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{\partial u_x}{\partial x} - \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} & \frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x} \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} & \frac{\partial u_y}{\partial y} - \frac{\partial u_y}{\partial y} & \frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y} \\ \frac{\partial u_z}{\partial \mathrm{x}} - \frac{\partial u_x}{\partial z} & \frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z} & \frac{\partial u_z}{\partial z} - \frac{\partial u_z}{\partial z} \end{pmatrix}$$

(Note that the on-diagonal elements of the anti-symmetric matrix are zero.)

In much more compact form:

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

$$= \varepsilon_{ij} + O_{ij}$$

Here the ε_{ij} terms ($= \frac{1}{2}\epsilon_{ij,in\ kittel}$) are related to elastic deformations. This is what we care about! Without giving any proof for this, I'll tell you that the O_{ij} terms are related to rigid rotations, and we'll disregard them as unimportant.

In Summary:

- $u_i(\mathbf{r}) - u_i(\mathbf{r}_0) = \sum_{j=1}^{3} \varepsilon_{ij}\,(r_j - r_{0j})$
- $\Delta\mathbf{u} = (\varepsilon_{ij}) \cdot (\Delta\mathbf{r})$ using matrix multiplication

Applications

1) What happens to the vector $a\hat{\mathbf{x}}$? Let $\mathbf{r} = a\hat{\mathbf{x}}, \mathbf{r}_0 = 0$

$$\Delta\mathbf{u} = \begin{pmatrix} & & \\ & \varepsilon_{ij} & \\ & & \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11}a \\ \varepsilon_{12}a \\ \varepsilon_{13}a \end{pmatrix}$$

The new vector is then,

$$\begin{aligned} new\ a\hat{\mathbf{x}} &= a\hat{\mathbf{x}} + a(\varepsilon_{11}\hat{\mathbf{x}} + \varepsilon_{12}\hat{\mathbf{y}} + \varepsilon_{13}\hat{\mathbf{z}}) \\ &= a[(1+\varepsilon_{11})\hat{\mathbf{x}} + \varepsilon_{12}\hat{\mathbf{y}} + \varepsilon_{13}\hat{\mathbf{z}}] \end{aligned}$$

This clearly has a different direction than $a\hat{\mathbf{x}}$. What about length?

$$\begin{aligned} \text{new length} &= [(new\ \ a\hat{\mathbf{x}}) \cdot (new\ \ a\hat{\mathbf{x}})]^{1/2} \\ &= a\sqrt{(1+\varepsilon_{11})^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2} \\ &= a(1 + 2\varepsilon_{11} + \varepsilon_{11}^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2)^{1/2} \end{aligned}$$

Keeping only terms which include one factor of ε in them (because the matrix values are assumed to be small), we have:

$$\text{new length} = a(1+\varepsilon_{11})$$

ε_{11} is therefore equal to the fractional change in length, namely $\varepsilon_{11} = \frac{\Delta L}{L} = \frac{new\ length - a}{a}$, which is precisely what we would normally call the "strain" for compression in the $\hat{\mathbf{x}}$ direction.

Similarly, ε_{22} and ε_{33} are equal to the strains in the $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ directions.

2) What happens to the angle between $a\hat{\mathbf{x}}$, and $a\hat{\mathbf{y}}$ (initially 90 degrees)? As done in Application 1,

$$new\ a\hat{\mathbf{x}} = a[(1+\varepsilon_{11})\hat{\mathbf{x}} + \varepsilon_{12}\hat{\mathbf{y}} + \varepsilon_{13}\hat{\mathbf{z}}]$$

Very similar for y,

$$new\ a\hat{\mathbf{y}} = a[\varepsilon_{21}\hat{\mathbf{y}} + (1+\varepsilon_{22})\hat{\mathbf{y}} + \varepsilon_{23}\hat{\mathbf{z}}]$$

From the dot product formula $(new\ a\hat{\mathbf{x}}) \cdot (new\ a\hat{\mathbf{y}}) = |new\ a\hat{\mathbf{x}}||new\ a\hat{\mathbf{y}}|\cos\theta$ ...

$$a^2[(1+\varepsilon_{11})(\varepsilon_{21}) + (\varepsilon_{12})(1+\varepsilon_{22}) + (\varepsilon_{13})(\varepsilon_{23})] = a^2(1+\varepsilon_{11})(1+\varepsilon_{22})\cos\theta$$

$$\cos\theta = [(1+\varepsilon_{11})(\varepsilon_{21}) + (\varepsilon_{12})(1+\varepsilon_{22}) + (\varepsilon_{13})(\varepsilon_{23})](1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{11}\varepsilon_{22})^{-1}$$

Doing a Taylor series expansion on the last term, namely $(1+x)^n \approx 1 + nx$, multiplying everything out, again keeping only terms which include one factor of ε in them, and using the symmetry of $\varepsilon_{21} = \varepsilon_{12}$, this greatly simplifies to

$$\cos\theta = \varepsilon_{21} + \varepsilon_{12} = 2\varepsilon_{12}$$

Since $\cos\theta$ is not zero, these two vectors are no longer perpendicular! And the off-diagonal terms like ε_{12} are a measure of how non-perpendicular the original coordinate axes are, after the deformation.

This is basically Kittel Eq. 3.32: $\mathbf{x}' \cdot \mathbf{y}' = \epsilon_{xy}$ (which =2 × our ε_{xy})

3) What happens to the volume of a cube? I'll skip the work here, but if you start with $(a\hat{\mathbf{x}}, a\hat{\mathbf{y}}, a\hat{\mathbf{z}})$, transform each vector, then calculate

$$\text{new volume} = (new\ a\hat{\mathbf{x}}) \cdot (new\ a\hat{\mathbf{y}} \times new\ a\hat{\mathbf{z}})$$

then make similar approximations as in Application 2, you arrive at:

$$\text{new volume} = a^3(1 + \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33})$$

Since the "trace" of a matrix is the sum of the diagonal elements, you can write:

$$\begin{aligned}
\text{new volume} &= a^3 \left(1 + \text{ Trace } \left(\varepsilon_{ij}\right)\right) \\
\frac{\text{new volume}}{\text{old volume}} &= 1 + \text{ Trace } \left(\varepsilon_{ij}\right) \\
\text{Trace } \left(\varepsilon_{ij}\right) &= \frac{\text{new volume}}{\text{old volume}} - 1 \\
\text{Trace } \left(\varepsilon_{ij}\right) &= \frac{\text{new volume} - \text{old volume}}{\text{old volume}} \\
\text{Trace } (\varepsilon_{ij}) &= \frac{\Delta V}{V}
\end{aligned}$$

So the trace of the ε_{ij} matrix relates to the fraction change in volume, if e.g. you compress the object on all sides as opposed to applying a stress only in one direction.

We will learn how to calculate ε_{ij} given the forces on the solid, and now you have some insight as to what the components of ε_{ij} mean.