

Debye Model for 3D - Summary by Dr. Colton

(a) Density of states $D(\omega) = \frac{1}{(2\pi/L)^3} \cdot 4\pi k^2 \cdot \frac{1}{v}$

$$= \frac{V}{8\pi^3} \cdot 4\pi \left(\frac{\omega}{v}\right)^2 \cdot \frac{1}{v}$$

$$D(\omega) = \frac{V}{2\pi^2 v^3} \omega^2$$

(b) cutoff frequency

$$N = \int_0^{\omega_D} D(\omega) d\omega$$

$$= \int_0^{\omega_D} \frac{V}{2\pi^2 v^3} \omega^2 d\omega$$

$$= \frac{V}{2\pi^2 v^3} \frac{1}{3} \omega_D^3$$

$$\omega_D = \left(6\pi^2 v^3 \frac{N}{V}\right)^{1/3}$$

(c) energy

$$U = 3 \times \int_0^{\omega_D} D(\omega) f(\omega) \hbar \omega d\omega$$

↑
of acoustic branches

$$= 3 \cdot \int_0^{\omega_D} \left(\frac{V}{2\pi^2 v^3} \omega^2\right) \left(\frac{1}{e^{\hbar\omega/kT} - 1}\right) \hbar \omega d\omega$$

$$\text{let } x = \frac{\hbar\omega}{kT} \rightarrow \omega = \frac{kT}{\hbar} x$$

$$(x_D = \frac{\hbar\omega_D}{kT}) \quad d\omega = \frac{kT}{\hbar} dx$$

$$= 3 \int_0^{x_D} \left[\frac{V}{2\pi^2 v^3} \left(\frac{kT}{\hbar} x\right)^2\right] \left(\frac{1}{e^x - 1}\right) \hbar \left(\frac{kT}{\hbar} x\right) \left(\frac{kT}{\hbar} dx\right)$$

$$U = \frac{3V}{2\pi^2 v^3} \frac{k^4 T^4}{\hbar^3} \int_0^{x_D} \frac{x^3}{e^x - 1} dx$$

(d) low T: $x_D \rightarrow \infty$

$$U = \frac{3V}{2\pi^2 v^3} \frac{k^4 T^4}{\hbar^3} \int_0^{\infty} \frac{x^3}{e^x - 1} dx$$

$\frac{\pi^4}{15}$ (via Mathematica)

3D cont

(d) low T, cont.

$$U = \frac{\pi^2}{10} \frac{V}{v^3} \frac{k^4}{h^3} T^4$$

$$C_V = \frac{\partial U}{\partial T} = \frac{\pi^2}{10} \frac{V}{v^3} \frac{k^4}{h^3} (4T^3)$$

$$C_V = \frac{2\pi^2}{5} \frac{k^4}{h^3} T^3 V$$

(can put in terms of N
and $\theta_D = \frac{h\nu_D}{k}$,
if desired)

(e) high T: $x = \text{small} \rightarrow e^x - 1 \approx (1+x) - 1 = x$

$$U = \frac{3}{2} \frac{V}{\pi^2 v^3} \frac{k^4 T^4}{h^3} \int_0^{x_D} \frac{x^3}{x} dx$$

$$\frac{1}{3} x_D^3$$

$$= \frac{1}{3} \frac{h^3 \omega_D^3}{k^3 T^3}$$

$$= \frac{1}{3} \frac{h^3}{k^3 T^3} \left(\frac{6\pi^2 v^3 N}{V} \right)$$

$$U = \frac{3}{2} \frac{V}{\pi^2 v^3} \frac{k^4 T^4}{h^3} \left[\frac{1}{3} \frac{h^3}{k^3 T^3} \frac{6\pi^2 v^3 N}{V} \right]$$

$$U = 3kTN$$

$$C_V = \frac{dU}{dT} = \boxed{3kN}$$

This matches the
Dulong-Petit law!
(and the equipartition theorem,
given 6 degrees of freedom)