

# Heat Capacity of 3D Electron Gas - Summary by Dr. Colton

(a) Density of States

$$D(E) = \frac{1}{(2\pi/L)^3} (4\pi k^2) \frac{1}{dE/dk} \times 2 \quad \text{spin states}$$

$$\text{or } E = \frac{\hbar^2 k^2}{2m} \rightarrow k = \frac{1}{\hbar} \sqrt{2mE}$$

$$\begin{aligned} \frac{dE}{dk} &= \frac{\hbar^2 k}{m} = \frac{\hbar^2}{m} \left( \frac{1}{\hbar} \sqrt{2mE} \right) \\ &= \hbar \sqrt{\frac{2E}{m}} \end{aligned}$$

$$D(E) = \frac{V}{8\pi^3} \left[ 4\pi \left( \frac{2mE}{\hbar^2} \right) \right] \left[ \frac{1}{\hbar \sqrt{\frac{2E}{m}}} \right] \quad (2)$$

$$D(E) = \frac{V}{2\pi^3} \left( \frac{2m}{\hbar^2} \right)^{3/2} E^{1/2} \quad (1)$$

(b) Fermi Energy

$$N = \int_0^{\infty} D(E) f(E) dE$$

$$f(E) = \begin{cases} 1 & E < E_F \\ 0 & E > E_F \end{cases} \quad \text{at 0 deg Kelvin}$$

$$N = \int_0^{E_F} D(E) dE$$

$$= \frac{V}{2\pi^3} \left( \frac{2m}{\hbar^2} \right)^{3/2} \underbrace{\int_0^{E_F} E^{1/2} dE}_{\frac{E_F^{3/2}}{3/2}}$$

$$E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3} \quad (2)$$

Alternate derivation:

$$\text{# states} = \text{volume} \times \frac{\text{\# states}}{\text{Volume}} \times 2$$

$$N = \left( \frac{4}{3}\pi k_F^3 \right) \frac{1}{(2\pi/L)^3} \times 2$$

$$N = \frac{V}{3\pi^2} k_F^3$$

$$= \frac{V}{3\pi^2} \left( \frac{1}{\hbar} \sqrt{2mE_F} \right)^3$$

$$E_F = \frac{\hbar^2}{2m} \left( \frac{3\pi^2 N}{V} \right)^{2/3} \quad \checkmark$$

(c) Calculate  $D(E_F)$  ... solve Eqn (2) for  $V$  and plug into Eqn (1)

$$\left(\frac{2mE_F}{\hbar^2}\right)^{3/2} = \frac{3\pi^2 N}{V}$$

$$V = 3\pi^2 N \left(\frac{\hbar^2}{2mE_F}\right)^{3/2}$$

then  $D(E_F) = \left[ 3\pi^2 N \left(\frac{\hbar^2}{2mE_F}\right)^{3/2} \right] \frac{1}{2\pi^3} \left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{1/2}$

$$D(E_F) = \frac{3}{2} \frac{N}{E_F}$$

(d) Energy:  $U = \int_0^\infty D(E) f(E) E dE$

$\downarrow$  approximate this as  $D(E_F)$

$\frac{1}{e^{(E-E_F)/kT} + 1}$

$\uparrow$  approximate  $\mu$  as  $E_F$ , a constant

$$U \approx D(E_F) \int_0^\infty \frac{E dE}{e^{(E-E_F)/kT} + 1}$$

(e) heat capacity:  $C_V = \frac{dU}{dT} \rightarrow$  take  $\frac{d}{dT}$  inside the integral

$$\frac{d}{dT} \left( \frac{1}{e^{(E-E_F)/kT} + 1} \right) = \frac{-1}{\left( e^{(E-E_F)/kT} + 1 \right)^2} \left( e^{-(E-E_F)/kT} \right) \left( \frac{E-E_F}{k} \right) \left( \frac{-1}{T^2} \right)$$

$$C_V = D(E_F) \frac{1}{kT^2} \int_0^\infty \frac{(E-E_F) E e^{-(E-E_F)/kT}}{\left( e^{(E-E_F)/kT} + 1 \right)^2} dE$$

let  $x = \frac{E-E_F}{kT} \rightarrow E = kTx + E_F$   
 $dE = kT dx$

$$C_V = D(E_F) k \int_{-E_F/kT}^\infty \frac{x e^x}{(e^x + 1)^2} (kTx + E_F) dx$$

(f) small T approx: take integral from  $-\infty$  to  $\infty$ , then  $\nearrow$  this part integrates to 0 since  $\frac{x e^x}{(e^x + 1)^2} = \text{odd}$

$$C_V = D(E_F) k^2 T \int_{-\infty}^\infty \frac{x^2 e^x}{(e^x + 1)^2} dx$$

$\pi^2/3$

$$C_V = \left( \frac{3}{2} \frac{N}{E_F} \right) (k^2 T) \left( \frac{\pi^2}{3} \right)$$

$$C_V = \frac{\pi^2}{2} \frac{k^2 T}{E_F} N$$

the contribution to the heat capacity from the electron gas.