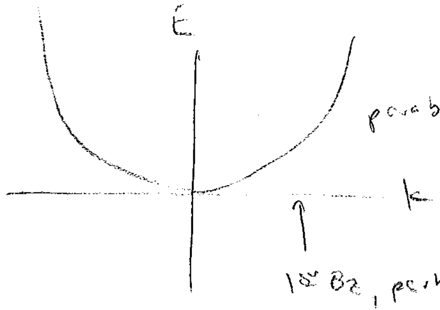
But note $\sin \frac{n\pi x}{L}$ $\searrow$ $k = \pm \frac{n\pi}{L}$

$$\boxed{E = \frac{\hbar^2 k^2}{2m}}$$

Compare $E = \frac{1}{2} m v^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m}$

$$\boxed{p = \hbar k}$$

same as momentum of wave



Note: for phonons dispersion curve repeated because of lattice periodicity

Here we haven't made that assumption yet. (But we will!)

(And same result!)

↓

- periodic dispersion,
- finite points separated by $2\pi/L$ (crystal dimension)

Density of States

3-D

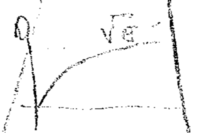
$$D(E) = \left(\underset{\text{spin states}}{\underset{\uparrow}{2}}\right) \frac{4\pi k^2}{dE/dk} \cdot \frac{1}{\left(\frac{2\pi}{L}\right)^3}$$

$$E = \frac{\hbar^2 k^2}{2m} \rightarrow \frac{dE}{dk} = \frac{2\hbar^2 k}{2m} = \frac{\hbar^2}{m}\sqrt{\frac{2mE}{\hbar^2}}$$

$$D(E) = \frac{2 \cdot 4\pi \left(\frac{2mE}{\hbar^2}\right)}{\frac{\hbar^2}{m}\sqrt{\frac{2mE}{\hbar^2}}} \times \frac{V}{(2\pi)^3}$$

$$\boxed{D(E) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}}$$

depends on next pg



factors of 2: $\frac{16}{\sqrt{2}\,8} = \frac{2}{\sqrt{2}} = \sqrt{2}$ // $\frac{2^{3/2}}{2} = 2^{1/2} = \sqrt{2}$ ✓ yes, the same

Goal: calculate heat capacity of electron gas

$$U = \int_0^\infty D(E) \; f(E) \; E \; dE$$

$D(E)$ → density of states

$f(E)$ → probability of state being occupied

E → energy of state

Same as for phonons except we'll use E instead of ω

then $C_V = \frac{\partial U}{\partial T}$ Compare: $Q = C_V \Delta T$ (for constant volume, no work, so $\Delta E = Q + \cancel{W_{on}}^{0}$)

Step 1: understand $f(E)$ a little better

$f(E)$ → Fermi Dirac distribution

Step 2: calculate $D(E)$ for electrons

(review how we did it for eg. acoustic phonons)

Step 3. piece together

Fermi-Dirac Distribution in detail

$$f = n_{ave} = \frac{1}{e^{(E-\mu)/kT}+1}$$

↓ how many particles at energy E

μ = chemical potential, energy needed to add a particle

= 0 for photons but

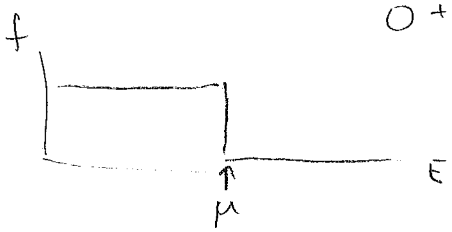
= important for electrons

$T=0$:
$$f = \frac{1}{e^{(E-\mu)/kT}+1} = \text{(if exponent = positive, } E \geq \mu\text{)}$$

$$\frac{1}{\text{huge}+1} = 0$$

$$= \text{(if exponent is negative, } E < \mu\text{)}$$

$$\frac{1}{0+1} = 1$$



empty [

full [

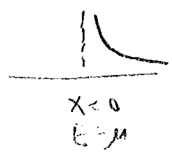
$\mu(T=0)$ called "Fermi Energy", E_F

Either a state is occupied, or it's not

T = small but not zero

if $E > \mu$, $\frac{1}{\text{huge}+1} = \frac{1}{\text{huge}} = \frac{1}{e^{(E-\mu)/kT}} = e^{-(E-\mu)/kT}$

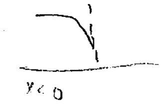
$= e^{-x/kT}$

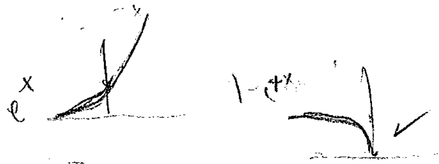


if $E < \mu$ $\frac{1}{\text{small}+1} = (1+\text{small})^{-1}$

$= 1 - \text{small}$

$= 1 - e^{+x/kT}$





at $E=\mu$ predicts 1

At $E=\mu$ predicts 0

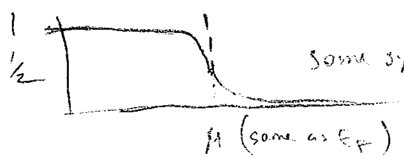
Actually: $f = \frac{1}{2}$

all empty [

some prob. of being full [

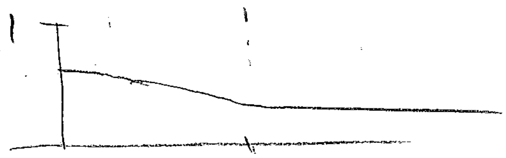
all full [

put together



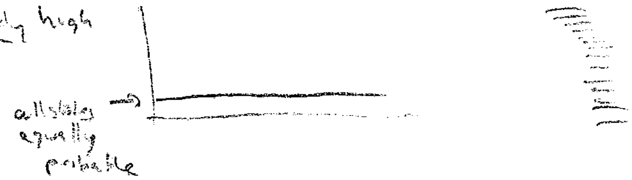
some symmetry... total area the same

T = high



very long tail
drags low E state probability down

T = extremely high



Note: when $(E-\mu) \gg k_B T$ ⟶ get Boltzmann statistics again

DOS review – what we did for phonons

$$D(\omega) = \underbrace{3}_{\text{long. + 2 transverse}} \times \underbrace{4\pi k^2}_{\text{shell model}} \times \left|\frac{1}{\frac{d\omega}{dk}}\right| \times \underbrace{\frac{1}{\left(\frac{2\pi}{L}\right)^3}}_{\text{\# pts in k-space volume}}$$

only in energy expression not in $D(\omega)$

Acoustic phonons: $\omega = vk$ approximation $\rightarrow k = \frac{\omega}{v}$

$d\omega = v\,dk \rightarrow \frac{d\omega}{dk} = v$

$$= \cancel{3} \times 4\pi\left(\frac{\omega}{v}\right)^2 \cdot \frac{1}{v} \cdot \frac{Vol}{(2\pi)^3}$$

Electrons: want an integral of E, not ω

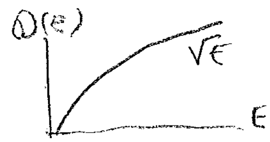
$$D(E) = \underbrace{2}_{\text{spin states}} \times 4\pi k^2 \times \frac{1}{\frac{dE}{dk}} \times \frac{1}{\left(\frac{2\pi}{L}\right)^3}$$

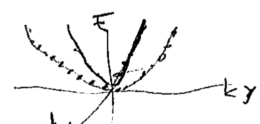
$$E = \frac{\hbar^2 k^2}{2m} \longrightarrow k^2 = \frac{2mE}{\hbar^2}$$

$$dE = \frac{\hbar^2 k\, dk}{m} \rightarrow \frac{dE}{dk} = \frac{\hbar^2 k}{m} = \frac{\hbar^2}{m}\left(\sqrt{\frac{2mE}{\hbar^2}}\right)$$

$$D(E) = 2 \cdot 4\pi\left(\frac{2mE}{\hbar^2}\right)\frac{1}{\frac{\hbar^2}{m}\sqrt{\frac{2mE}{\hbar^2}}} \quad \frac{Vol}{(2\pi)^3}$$

$$\boxed{D(E) = \frac{Vol}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}}$$





2D can sort of visualize

3D hard

proof of factors of 2 in last step

$$\frac{2 \times 4 \times 2}{\sqrt{2} \times 2^3} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

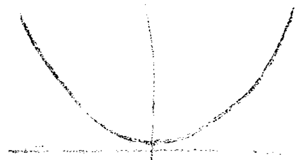
also $\frac{2^{3/2}}{2} = \sqrt{2}$

{ 1D and 2D for HW ✓ } 2D.

→ 1D $D(E)$ diverges (at origin?) 1D

day 21 pg 6

Back to $E = \frac{\hbar^2 k^2}{2m}$



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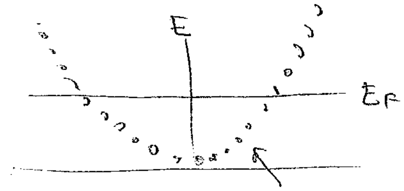

at T=0, all states below E_F are filled; none above

How to find μ:

$$N = \int_0^\infty D(E)\, f(E)\, dE$$

if only 1 particle/state, then f = average occupation also = probability of occupation!

$$N = \frac{Vol}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \sqrt{E}\,\frac{dE}{e^{(E-\mu)/kT}+1}$$

μ = only unknown, could solve numerically
(it can become negative, for large values of T!)

day 22 pg 1

0/13 data on next page

At T=0, $\mu = E_F$

$$N = \frac{Vol}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\underbrace{\int_0^\infty \sqrt{E}\, dE \times \begin{pmatrix} 1 & \text{if } E<E_F \\ 0 & \text{if } E>E_F\end{pmatrix}}_{\int_0^{E_F} E^{1/2}\, dE}$$

$$\left.\frac{E^{3/2}}{3/2}\right|_0^{E_F} = \frac{2}{3}E_F^{3/2}$$

$$\frac{N}{V} = \frac{1}{\cancel{2}\pi^2}\times\frac{\cancel{2}}{3}\times\left(\frac{2mE_F}{\hbar^2}\right)^{3/2}$$

$$\boxed{E_F = \left(\frac{3\pi^2 N}{V}\right)^{2/3}\times\frac{\hbar^2}{2m}}$$

for HW: extend to 1D and 2D (shouldn't be too hard)

Remember if T = small (but not zero), μ still $\simeq E_F$

Typical value of E_F

use sodium $n = \frac{N}{V} = 2.652 \cdot 10^{28}\ /m^3$ from Table 1.4 pg 21
(assume one electron / atom)

$m = 9.11\ 10^{-31}$ kg (actually mass acts like it's reduced) → more later

then $E_F = 3.23$ eV Compare $k_B T_{room} = 20$ meV
$k_F = 9.2 \cdot 10^7$ rad/cm $k_{Brillouin\ Zone} = \frac{\pi}{a} = 5.2 \cdot 10^7\ \frac{rad}{cm}$

∴ $\boxed{\text{temp = small! and all usable ks are in "Fermi sphere"}}$ or very close to it