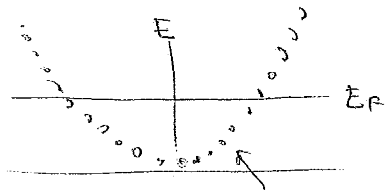
day 21 pg 6

Back to $E = \frac{\hbar^2 k^2}{2m}$  = 

at $T=0$, all states below E_F are filled; none above

How to find μ:

$$N = \int_0^\infty D(E)\, f(E)\, dE$$

if only 1 particle/state, then f = average occupation also = probability of occupation!

$$N = \frac{Vol}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \int_0^\infty \sqrt{E}\, \frac{dE}{e^{(E-\mu)/kT}+1}$$

μ = only unknown, could solve numerically (it can become negative, for large values of T!)

day 22 pg 1

At $T=0$, $\mu = E_F$

$$N = \frac{Vol}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \underbrace{\int_0^\infty \sqrt{E}\, dE \times \begin{pmatrix} 1 & \text{if } E<E_F \\ 0 & \text{if } E>E_F \end{pmatrix}}_{\int_0^{E_F} E^{1/2}\, dE}$$

$$\left.\frac{E^{3/2}}{3/2}\right|_0^{E_F} = \frac{2}{3} E_F^{3/2}$$

$$\frac{N}{V} = \frac{1}{\cancel{2}\pi^2} \times \frac{\cancel{2}}{3} \times \left(\frac{2mE_F}{\hbar^2}\right)^{3/2}$$

$$\boxed{E_F = \left(\frac{3\pi^2 N}{V}\right)^{2/3} \cdot \frac{\hbar^2}{2m}}$$

obtaining N & E_F

for HW: extend to 1D and 2D (shouldn't be too hard)

Remember if T = small (but not zero), μ still $\simeq E_F$

Typical value of E_F

use sodium $n = \frac{N}{V} = 2.652 \cdot 10^{28}\ /m^3$ from Table 1.4 pg 21 (assume one electron/atom)

$m = 9.11\ 10^{-31}$ kg (actually mass acts like it's reduced) → more later

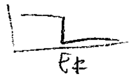
then $E_F = 3.23$ eV — Compare $k_B T_{room} = 20$ meV

$k_F = 9.2 \cdot 10^{7}$ rad/cm — $k_{Brillouin\ Zone} = \frac{\pi}{a} = 5.2 \cdot 10^{7}$ rad/cm

$\therefore$ T_{emp} = small! and all usable ks are in "Fermi sphere" or very close to it

Calculating the Fermi Energy

$$N = \int_0^\infty D(E)\, f(E)\, dE$$

at 0K $f(E) =$ 

$$= \int_0^{E_F} D(E)\, dE$$

$$= \int_0^{E_F} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E}\, dE$$

$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \left.\frac{E^{3/2}}{3/2}\right|_0^{E_F}$$

$$N = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2}$$

$$E_F = \left[3\pi^2 \frac{N}{V}\left(\frac{\hbar^2}{2m}\right)^{3/2}\right]^{2/3}$$

$$\boxed{E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}}$$

Easier way?: do it in k-space

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m}$$

→ Volume × $\frac{\#'s}{vol}$ = #'s

× 2 (spin states)

$$N = \frac{4}{3}\pi k_F^3 \cdot \frac{1}{(2\pi/L)^3} \cdot 2 = \frac{4}{3}\pi \frac{V}{8\pi^3} 2 k_F^3$$

$$= \frac{1}{3\pi^2} V k_F^3$$

$$k_F = \left(3\pi^2 \cdot \frac{N}{V}\right)^{1/3}$$

$$\boxed{E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}}$$ same!

$$\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\left[\frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}\right]^{1/2}$$

$$\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\frac{\hbar}{\sqrt{2m}}\left(3\pi^2\frac{N}{V}\right)^{1/3}$$

$$\frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{1/2} \stackrel{?}{=} \frac{3N}{2E_F}$$

Also $E_F = \left(\frac{\hbar^2}{2m}\right)^{3/2}$

$$D(E_F) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{1/2} \longrightarrow \text{want to simplify}$$

Also $E_F = \frac{\hbar^2}{2m}\left(3\pi^2\frac{N}{V}\right)^{2/3}$ — from yesterday or earlier today

$\downarrow$ solve for V

$$\left(\frac{2mE_F}{\hbar^2}\right)^{3/2} = 3\pi^2\frac{N}{V}$$

$$V = 3\pi^2 N\left(\frac{2mE_F}{\hbar^2}\right)^{-3/2}$$

$$D(E_F) = \frac{\left[3\cancel{\pi^2}N\left(\frac{\cancel{2m}E_F}{\cancel{\hbar^2}}\right)^{-3/2}\right]}{2\cancel{\pi^2}}\left(\frac{\cancel{2m}}{\cancel{\hbar^2}}\right)^{3/2} E_F^{1/2}$$

$$\boxed{D(E_F) = \frac{3}{2}\frac{N}{E_F}}$$

useful simplified form as you'll see in a minute

@ Back to Heat Capacity of metals calculation

$$U = \int_0^\infty D(\epsilon)\, f(\epsilon) \times E \; dE$$ weighted average of energies

$$C_V = \frac{dU}{dT} = \int_0^\infty D(E) \left(\frac{df}{dT}\right) E \; dE$$

(no work, so Q = U)

↓ really need to know temp dependence of μ, but let's say $\mu \simeq E_F$ (constant)

$$f \simeq \left[e^{(E-E_F)/kT} + 1 \right]^{-1}$$

$$\frac{df}{dT} = -1 \left[e^{(E-E_F)/kT} + 1 \right]^{-2} e^{(E-E_F)/kT} \left(\left(\frac{E-E_F}{k}\right)\left(-\frac{1}{T^2}\right) \right)$$

$$= \frac{e^{(E-E_F)/kT}}{\left(e^{(E-E_F)/kT} + 1\right)^2} \frac{E-E_F}{kT^2}$$

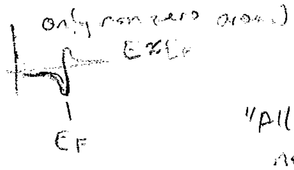
plug into integral

plug $D(E) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E}$ into integral

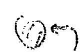
Note: Kittel uses $\tau = k_B T$

Result for "mess of math"? After approximation – use $D(E) \approx$ constant $= D(E_F)$ (take out of the integral)

$$C_V = D(E_F) \int_0^\infty \underbrace{\frac{e^{(E-E_F)/kT}}{\left(e^{(E-E_F)/kT} + 1\right)^2} \frac{(E-E_F)}{kT^2}}_{\frac{df}{dT}} E \; dE$$



"All action takes place near edge of Fermi sphere"



Students may skip most of this algebra...

Day 32 pg 5

let $x = \frac{E - E_F}{kT} \longrightarrow E = kTx + E_F$
$dE = kT\,dx$

$$C_v = D(E_F) \int_{-\frac{E_F}{kT}}^{\infty} \frac{e^x}{(e^x+1)^2} \cdot \frac{x}{T} \left(\cancel{kTx + E_F}\right)(kT\,dx)$$

if T small take this limit to $-\infty$

E_F integrates to zero because $\frac{xe^x}{(e^x+1)^2}$ odd

$$C_v = D(E_F)\,\frac{k\cancel{T}}{\cancel{T}} \cdot kT \underbrace{\int_{-\infty}^{\infty} \frac{x^2 e^x}{(e^x+1)^2}\,dx}_{\pi^2/3}$$

$$\boxed{C_v = \frac{\pi^2}{3} D(E_F)\,k^2 T}$$

could plug in for this if we wanted. Bottom line $C_v \sim T$

summary handout

~~next page: simplified form of $D(E_F) = \frac{3}{2}\frac{N}{E_F}$~~

Low temp: $\boxed{C_{v\,tot} = \underbrace{\gamma T}_{electron} + \underbrace{AT^3}_{phonon}}$

Fig 9 pg 145

$$\frac{C_v}{T} = m(T^2) + b$$

C_v/T

potassium

fits perfectly!

T^2

Table 2 pg 146

Experimental γ vs Theoretical γ

→ pretty close, but some differences,

Attribute differences to "(thermal) effective mass"

Trace back... $C_v \sim D(E_F) \sim m^{3/2} E_F^{1/2}$
$\hookrightarrow E_F \sim \frac{1}{m}$ so $E_F^{1/2} \sim \frac{1}{m^{1/2}}$

$C_v \sim m$