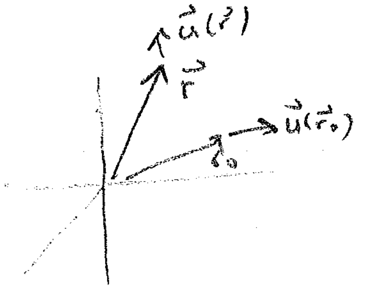
Day 10 pg 1

→ Hand out Exam

→ go over handout on Strain

including warnings about Kittel's notation

I found Kittel pp 73-75 to be nearly incomprehensible. After doing some research and talking to an acoustics professor, I found a much better description in Fetter and Walecka, Theoretical Mechanics of Particles and Continua, pp 460-463. This mostly follows their treatment.



$\vec{u}$ = displacement field, describes where each point goes to under a deformation.

I.e. $\vec{r}\,' = \vec{r} + \vec{u}(\vec{r})$ is new position of $\vec{r}$

$\vec{r}_0\,' = \vec{r}_0 + \vec{u}(\vec{r}_0)$ is new position of $\vec{r}_0$

Consider only small displacements → do a Taylor's series, 1st order

Taylor's Series Review:

— 1D : $f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0}(x - x_0)$

— 3D scalar function: $f(x,y,z) = f(x_0,y_0,z_0) + \frac{\partial f}{\partial x}(x-x_0) + \frac{\partial f}{\partial y}(y-y_0) + \frac{\partial f}{\partial z}(z-z_0)$

where each derivative is evaluated at (x_0,y_0,z_0)

equivalently: $f(\vec{r}) = f(\vec{r}_0) + \left.\vec{\nabla} f\right|_{\vec{r}_0} \cdot (\vec{r} - \vec{r}_0)$

equivalently: $f(\vec{r}) - f(\vec{r}_0) = \sum_{j=1}^{3} \left.\frac{\partial f}{\partial x_j}\right|_{\vec{r}_0} (r_j - r_{0j})$

where $j = 1,2,3$ refers to x,y,z

— 3D vector function: each component of $\vec{u}$ (u_x, u_y, u_z) is a scalar function

For the i^{th} component:

$$\underbrace{u_i(\vec{r}) - u_i(\vec{r}_0) = \sum_{j=1}^{3} \left.\frac{\partial u_i}{\partial x_j}\right|_{\vec{r}_0} (r_j - r_{0j})}_{\text{this is just like matrix multiplication}}$$

$$\begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \\ \frac{\partial u_y}{\partial x} & & \\ & \text{etc} & \end{pmatrix} \begin{pmatrix} r_x - r_{0x} \\ r_y - r_{0y} \\ r_z - r_{0z} \end{pmatrix}$$

The 3x3 matrix $\frac{\partial u_i}{\partial x_j}$ is called the "displacement gradient"

Break for some notation notes: $\vec{u}$ is the same as Kittel's $\vec{R}$, eqn. 27

u_x is what Kittel calls u
u_y ... v
u_z ... w

The nine $\frac{\partial u_i}{\partial x_j}$ components are Kittel's e_{ij} components,
↳ this is a non-standard symbol for the displacement gradient

Now write the displace. gradient as a sum of symm. and antisymm matrices

$$= \frac{1}{2}\begin{pmatrix} \frac{\partial u_x}{\partial x}+\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y}+\frac{\partial u_y}{\partial x} & \\ \frac{\partial u_y}{\partial x}+\frac{\partial u_x}{\partial y} & & \\ & \text{etc.} & \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{\partial u_x}{\partial x}-\frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y}-\frac{\partial u_y}{\partial x} & \\ \frac{\partial u_y}{\partial x}-\frac{\partial u_x}{\partial y} & & \\ & \text{etc.} & \end{pmatrix}$$

in much more compact form

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)}_{\text{call this the matrix } \varepsilon_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}-\frac{\partial u_j}{\partial x_i}\right)}_{\text{call this the matrix } O_{ij}}$$

ε_{ij} ($= \frac{1}{2}$ of Kittel's e_{ij})
↳ This is related to elastic deformations
↳ this is what we care about!

O_{ij}
↳ This is related to rigid rotations

Summary

$$\boxed{\begin{aligned} u_i(\vec{r})-u_i(\vec{r}_o) &= \sum_{j=1}^{3} \varepsilon_{ij}(r_j - r_{oj}) \\ \begin{pmatrix}\Delta u\end{pmatrix} &= \begin{pmatrix}\varepsilon_{ij}\end{pmatrix}\begin{pmatrix}\Delta r\end{pmatrix} \end{aligned}}$$ in matrix form

Applications

1) What happens to the vector $a\hat{x}$? Let $\vec{r} = a\hat{x}$, $\vec{r}_0 = 0$

$$\Delta\vec{u} = \begin{pmatrix} & & \\ & \varepsilon_{ij} & \\ & & \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} a \\ \varepsilon_{12} a \\ \varepsilon_{13} a \end{pmatrix}$$

$$\text{New vector} = a\hat{x} + a(\varepsilon_{11}\hat{x} + \varepsilon_{12}\hat{y} + \varepsilon_{13}\hat{z})$$
$$= a\left[(1+\varepsilon_{11})\hat{x} + \varepsilon_{12}\hat{y} + \varepsilon_{13}\hat{z}\right]$$

This clearly has a different direction than $a\hat{x}$.

What about length ?

$$a' = \text{new length} = a\sqrt{(1+\varepsilon_{11})^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2}$$
$$= a\left(1 + 2\varepsilon_{11} + \cancel{\varepsilon_{11}^2} + \cancel{\varepsilon_{12}^2} + \cancel{\varepsilon_{13}^2}\right)^{1/2}$$

keep only first order

$$\boxed{a' = a(1+\varepsilon_{11})}$$

$$\text{or } \varepsilon_{11} = \frac{a'}{a} - 1 = \frac{a'-a}{a} = \frac{\Delta L}{L} \; !$$

ε_{11} = Physics 121 strain for compression in $\hat{x}$

2) What happens to the angle between $a\hat{x}$ and $a\hat{y}$? (initially $90°$)

$$\text{New } a\hat{x} = a\left[(1+\varepsilon_{11})\hat{x} + \varepsilon_{12}\hat{y} + \varepsilon_{13}\hat{z}\right]$$

very similarly,

$$\text{New } a\hat{y} = a\left[\varepsilon_{21}\hat{x} + (1+\varepsilon_{22})\hat{y} + \varepsilon_{23}\hat{z}\right]$$

From $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ ...

$$\cancel{a^2}\left[(1+\varepsilon_{11})(\varepsilon_{21}) + (\varepsilon_{12})(1+\varepsilon_{22}) + (\varepsilon_{12})(\varepsilon_{23})\right] = \cancel{a^2}\cos\theta$$

again keeping only first order

$$\cos\theta = \varepsilon_{21} + \varepsilon_{12}$$ but ε_{ij} is symmetric, so

$$\boxed{\cos\theta = 2\varepsilon_{12}}$$

the two vectors are no longer perpendicular

Compare to Kittel Eqn 32: $\underbrace{\vec{x}'}_{\text{new }\hat{x}} \cdot \underbrace{\vec{y}'}_{\text{new }\hat{y}} = e_{xy}$

$\hookrightarrow 2 \times$ our ε_{12}

3) What happens to volume of a cube?

Start with $a\hat{x}$, $a\hat{y}$, $a\hat{z}$

Transform each vector

Calculate volume $= (\text{new } a\hat{x}) \cdot (\text{new } a\hat{y} \times \text{new } a\hat{z})$

keep only first order

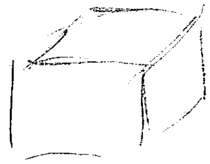
Result : $\boxed{\text{new Volume} = a^3 \left(1 + \underbrace{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}_{\text{trace of } \varepsilon_{ij} \text{ matrix}}\right)}$

or $\text{Trace}(\varepsilon_{ij}) = \frac{\Delta V}{V}$!

We will learn how to calculate ε_{ij} given the forces on the solid, and now you have some insight as to what the components of ε_{ij} mean.

day 10 pg 2

Generalized stress



σ

(not used in Kittel)

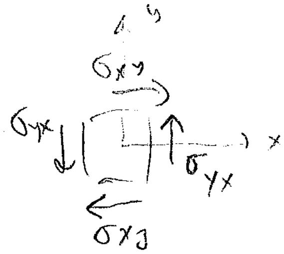
σ_{ij} = i ... j

compressive: σ_{xx} = force in x direction per area applied to area, normal in x-dir

shear: σ_{xy} = y y

Note: $\sigma_{xy} = \sigma_{yx}$

otherwise torque / ang. accel



σ_{ij}: symmetric matrix, 6 independent components

$$\begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ & \sigma_{yy} & \sigma_{yz} \\ '' & & \sigma_{zz} \end{pmatrix}$$

write as 6 component vector

$$\begin{pmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{zz} \\ \sigma_{yz} \\ \sigma_{xz} \\ \sigma_{xy} \end{pmatrix}$$

same thing w/ strain matrix: 6 component vector.

Kittel notation: $\sigma_{xy} \rightarrow X_y$ Derek Thomas: never seen before!

Relating stress to strain

$$\begin{pmatrix} e_1 \\ \vdots \\ e_6 \end{pmatrix} \text{ or } \begin{pmatrix} e_{xx} \\ \vdots \\ e_{xy} \end{pmatrix} = \begin{pmatrix} S \\ \text{"compliance tensor"} \end{pmatrix} \begin{pmatrix} \sigma_{xx} \\ \vdots \\ \sigma_{xy} \end{pmatrix} \text{ or } \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_6 \end{pmatrix}$$

to translate back, $S_{23} = S_{(yy)(zz)}$

$\uparrow$ pos. 2 $\quad \uparrow$ pos. 3

but only 36 components (potentially) instead of 81

S describes how easily substance is deformed

Inverse matrix

C = "stiffness tensor"

(they got the letters backwards! :) S = compliance C = stiffness)

$$\begin{pmatrix} \sigma \end{pmatrix} = \begin{pmatrix} C \end{pmatrix} \begin{pmatrix} e \end{pmatrix}$$

Note: we will return to Kittel's defn of e now

where $\overset{\text{Kittel's}}{e_{xy}} = 2 \times \overset{\text{more standard}}{\varepsilon_{xy}}$

I do this so we can use Kittel's eqns and tabulated values for components of C

Otherwise would have C_{44} values (next page) which are off by factor of 2.

cubic crystal use symmetry to figure out which elements are 0, and which are equal to others

consider $C_{14} = C_{xxyz}$

$$\begin{pmatrix}\sigma_{xx}\\ \vdots \\ \sigma_{xy}\end{pmatrix} = \left(\begin{array}{c|c} & c_{xy} \\ \hline & \end{array}\right)\begin{pmatrix}e_{xx}\\ \vdots \\ e_{xy}\end{pmatrix}$$

$$\sigma_{xx} = \sim + C_{14}\underbrace{e_{yz}}_{\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}} + \cdots$$

Reflection symmetry: $z \to -z$ $e_{yz} \to -e_{yz}$ because $\frac{\partial v}{\partial z} \to -\frac{\partial v}{\partial z}$ and $\frac{\partial w}{\partial y} \to -\frac{\partial w}{\partial y}$

perform operation:

$$\sigma_{xx} = \sim + C_{14}(-e_{yz}) + \cdots$$

can only be equal if $\boxed{C_{14} = 0}$

Similarly 3-fold rotation symmetry about [111] axis $x \to y$, $y \to z$, $z \to x$

forces $\underset{xxxx}{C_{11}} = \underset{yyyy}{C_{22}} = \underset{zzzz}{C_{33}}$ (call it C_{11})

and $C_{44} = C_{55} = C_{66}$

C_{yzyz} C_{xzxz} C_{xyxy}

$\downarrow$ by rotation

C_{zxzx} which = this

Result:

$$\boxed{C = \left(\begin{array}{ccc|ccc} C_{11} & C_{12} & C_{12} & & & \\ C_{12} & C_{11} & C_{12} & & 0 & \\ C_{12} & C_{12} & C_{11} & & & \\ \hline & & & C_{44} & & 0 \\ & 0 & & & C_{44} & \\ & & & 0 & & C_{44} \end{array}\right)}$$

only 3 independent components!

(down from 81!)

see tables 11+12, pg 84

for values of C_{11}, C_{12}, C_{44}