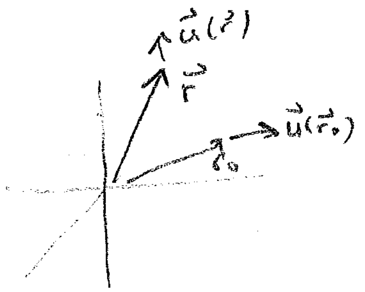
Elastic Strains - Phys 581 handout

I found Kittel pp 73-75 to be nearly incomprehensible. After doing some research and talking to an acoustics professor, I found a much better description in Fetter and Walecka, Theoretical Mechanics of Particles and Continua, pp 460-463. This mostly follows their treatment.



$\vec{u}$ = displacement field, describes where each point goes to under a deformation.

I.e. $\vec{r}\,' = \vec{r} + \vec{u}(\vec{r})$ is new position of $\vec{r}$

$\vec{r}_0\,' = \vec{r}_0 + \vec{u}(\vec{r}_0)$ is new position of $\vec{r}_0$

Consider only small displacements → do a Taylor's series, 1st order

Taylor's series Review:

— 1D : $f(x) = f(x_0) + \left.\frac{\partial f}{\partial x}\right|_{x_0} (x - x_0)$

— 3D scalar function: $f(x,y,z) = f(x_0,y_0,z_0) + \frac{\partial f}{\partial x}(x - x_0) + \frac{\partial f}{\partial y}(y - y_0) + \frac{\partial f}{\partial z}(z - z_0)$

where each derivative is evaluated at (x_0, y_0, z_0)

equivalently: $f(\vec{r}) = f(\vec{r}_0) + \left.\vec{\nabla} f\right|_{\vec{r}_0} \cdot (\vec{r} - \vec{r}_0)$

equivalently: $f(\vec{r}) - f(\vec{r}_0) = \sum_{j=1}^{3} \left.\frac{\partial f}{\partial x_j}\right|_{\vec{r}_0} (r_j - r_{0j})$

where $j = 1,2,3$ refers to x, y, z

— 3D vector function: each component of $\vec{u}$ (u_x, u_y, u_z) is a scalar function

for the i^{th} component:

$$u_i(\vec{r}) - u_i(\vec{r}_0) = \underbrace{\sum_{j=1}^{3} \left.\frac{\partial u_i}{\partial x_j}\right|_{\vec{r}_0} (r_j - r_{0j})}_{\text{this is just like matrix multiplication}}$$

$$\begin{pmatrix} \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} & \\ \frac{\partial u_y}{\partial x} & \text{etc} & \\ & & \end{pmatrix} \begin{pmatrix} r_x - r_{0x} \\ r_y - r_{0y} \\ r_z - r_{0z} \end{pmatrix}$$

The 3x3 matrix $\frac{\partial u_i}{\partial x_j}$ is called the "deformation gradient"

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Break for some notation notes: $\vec{u}$ is the same as Kittel's $\vec{R}$, eqn. 27

u_x is what Kittel calls u

u_y $\quad v$

u_z $\quad w$

The nine $\frac{\partial u_i}{\partial x_j}$ components are Kittel's ϵ_{ij} components,

↳ this is a non-standard symbol for the deformation gradient

Now write the deform. gradient as a sum of symm. and antisymm. matrices

$$= \frac{1}{2}\begin{pmatrix} \frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} & \\ \frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y} & & \\ & \text{etc.} & \end{pmatrix} + \frac{1}{2}\begin{pmatrix} \frac{\partial u_x}{\partial x} - \frac{\partial u_x}{\partial x} & \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} & \\ \frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y} & & \\ & \text{etc.} & \end{pmatrix}$$

in much more compact form

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}_{\text{call this the matrix } \varepsilon_{ij}} + \underbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}_{\text{call this the matrix } O_{ij}}$$

($= \frac{1}{2}$ of Kittel's e_{ij})

↳ This is related to elastic deformations

↳ this is what we care about!

↳ This is related to rigid rotations

Summary

$$\boxed{\begin{gathered} u_i(\vec{r}) - u_i(\vec{r}_0) = \sum_{j=1}^{3} \varepsilon_{ij}(r_j - r_{0j}) \\ \begin{pmatrix} \Delta u \\ \\ \end{pmatrix} = \begin{pmatrix} & \varepsilon_{ij} & \\ & & \end{pmatrix}\begin{pmatrix} \Delta r \\ \\ \end{pmatrix} \end{gathered}}$$

in matrix form

Applications

1) What happens to the vector $a\hat{x}$? Let $\vec{r} = a\hat{x}$, $\vec{r}_0 = 0$

$$\Delta\vec{u} = \begin{pmatrix} & & \\ & \varepsilon_{ij} & \\ & & \end{pmatrix}\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \varepsilon_{11} a \\ \varepsilon_{12} a \\ \varepsilon_{13} a \end{pmatrix}$$

$$\text{New vector} = a\hat{x} + a\left(\varepsilon_{11}\hat{x} + \varepsilon_{12}\hat{y} + \varepsilon_{13}\hat{z}\right)$$
$$= a\left[(1+\varepsilon_{11})\hat{x} + \varepsilon_{12}\hat{y} + \varepsilon_{13}\hat{z}\right]$$

This clearly has a different direction than $a\hat{x}$.

What about length?

$$a' = \text{new length} = a\sqrt{(1+\varepsilon_{11})^2 + \varepsilon_{12}^2 + \varepsilon_{13}^2}$$
$$= a\left(1 + 2\varepsilon_{11} + \cancel{\varepsilon_{11}^2} + \cancel{\varepsilon_{12}^2} + \cancel{\varepsilon_{13}^2}\right)^{1/2}$$

keep only first order

$$\boxed{a' = a(1+\varepsilon_{11})}$$

$$\text{or } \varepsilon_{11} = \frac{a'}{a} - 1 = \frac{a'-a}{a} = \frac{\Delta L}{L}\ !$$

ε_{11} = Physics 121 strain for compression in $\hat{x}$

2) What happens to the angle between $a\hat{x}$ and $a\hat{y}$? (initially 90°)

$$\text{New } a\hat{x} = a\left[(1+\varepsilon_{11})\hat{x} + \varepsilon_{12}\hat{y} + \varepsilon_{13}\hat{z}\right]$$

very similarly,

$$\text{New } a\hat{y} = a\left[\varepsilon_{21}\hat{x} + (1+\varepsilon_{22})\hat{y} + \varepsilon_{23}\hat{z}\right]$$

From $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$ ...

$$\cancel{a^2}\left[(1+\varepsilon_{11})(\varepsilon_{21}) + (\varepsilon_{12})(1+\varepsilon_{22}) + (\varepsilon_{12})(\varepsilon_{23})\right] = \cancel{a^2}\cos\theta$$

again keeping only first order

$$\cos\theta = \varepsilon_{21} + \varepsilon_{12}$$ but ε_{ij} is symmetric, so

$$\boxed{\cos\theta = 2\varepsilon_{12}}$$ the two vectors are no longer perpendicular

Compare to Kittel Eqn 32: $\underbrace{\vec{x}'}_{\text{new } \hat{x}} \cdot \underbrace{\vec{y}'}_{\text{new } \hat{y}} = e_{xy}$ ↳ $2 \times$ our ε_{12}

3) What happens to volume of a cube?

Start with $a\hat{x}$, $a\hat{y}$, $a\hat{z}$

Transform each vector

Calculate volume = $(\text{new } a\hat{x}) \cdot (\text{new } a\hat{y} \times \text{new } a\hat{z})$

keep only first order

Result : $\boxed{\text{new Volume} = a^3 (1 + \underbrace{\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}}_{\text{trace of } \varepsilon_{ij} \text{ matrix}})}$

or $\text{Trace}(\varepsilon_{ij}) = \frac{\Delta V}{V}$!

We will learn how to calculate ε_{ij} given the forces on the solid, and now you have some insight as to what the components of ε_{ij} mean.