How to Solve for wave speed in a given direction - Phys 581

Wave information

1) General exponential dependence is $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$
2) Direction of travel tells you what $\vec{k}$ components are present
3) Direction of oscillation (ie, longitudinal vs transverse, and which specific transverse direction) tells you which $\vec{u}$ components are present.

How to use Eqn 3.57 (a), (b), (c).

1) The Easy Way - If you know the specifics of u_x, u_y, u_z, Plug that in from the start.

Example 1 - (100) waves, longitudinal

Given $\vec{k} = (k,0,0)$ and $\vec{u} = u_x \hat{x} e^{i(kx-\omega t)}$

Eqn 3.57 a becomes

$$\rho\omega^2 \cancel{x} = C_{11} k^2 \cancel{x} \rightarrow \boxed{v = \frac{\omega}{k} = \sqrt{\frac{C_{11}}{\rho}}}$$

Example 2 - (100) waves, transverse in the (0,1,0) direction

Given $\vec{k} = (k,0,0)$ and $u = (0,u,0)$

I.e. $\vec{u} = u_y \hat{y} e^{i(kx-\omega t)}$

Eqn 3.57 a becomes

$$\rho\omega^2 \cancel{x} = C_{44} k^2 \cancel{x} \rightarrow \boxed{\frac{\omega}{k} = \sqrt{\frac{C_{44}}{\rho}}}$$

Example 3: (110) waves, longitudinal

Given $\vec{k} = k\frac{(\hat{x}+\hat{y})}{\sqrt{2}}$ and $\vec{u} = u\frac{(\hat{x}+\hat{y})}{\sqrt{2}} e^{i\left(k\left(\frac{x+y}{\sqrt{2}}\right)-\omega t\right)}$

Eqn 3.57a becomes

$$\rho\omega^2 u = C_{11}\frac{k^2}{2}u + C_{44}\frac{k^2}{2}u + C_{12}+C_{44}\left(\frac{k^2}{2}u\right)$$

$$\rightarrow \boxed{\omega/k = \sqrt{\frac{1}{2\rho}(C_{11}+C_{12}+2C_{44})}}$$

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2) The Hard Way – if you don't know all of the specifics, you'll need to solve simultaneous eqns in an eigenvalue-type matrix eqn.

Example 4: (110) waves, unknown direction in x-y plane

Given $\vec{k} = k\left(\frac{\hat{x}+\hat{y}}{\sqrt{2}}\right)$, $\vec{u} = \underbrace{\left(u_x\hat{x}+u_y\hat{y}\right)}_{\text{these unknown. But } u_z = 0 \text{ is known.}} e^{i\left(k\left(\frac{x+y}{\sqrt{2}}\right)-\omega t\right)}$

this still known because of k

Eqn 3.57a: $\rho\omega^2 u_x = C_{11}\left(\frac{k^2}{2}\right)u_x + C_{44}\left(\frac{k^2}{2}\right)u_x + \left(C_{12}+C_{44}\right)\left(\frac{k^2}{2}\right)u_y$

or $\rho\omega^2 u_x = \frac{1}{2}k^2\left(C_{11}+C_{44}\right)u_x + \frac{1}{2}k^3\left(C_{12}+C_{44}\right)u_y$

Eqn 3.57b: $\rho\omega^2 u_y = \frac{1}{2}k^2\left(C_{11}+C_{44}\right)u_y + \frac{1}{2}k^2\left(C_{12}+C_{44}\right)u_x$

Combine in matrix eqn:

$$\begin{pmatrix} \rho\omega^2 & 0 \\ 0 & \rho\omega^2 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} = \frac{1}{2}k^2\begin{pmatrix} C_{11}+C_{44} & C_{12}+C_{44} \\ C_{12}+C_{44} & C_{11}+C_{44} \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix}$$

$$\begin{pmatrix} \frac{k^2}{2}\left(C_{11}+C_{44}\right)-\rho\omega^2 & \frac{k^2}{2}\left(C_{12}+C_{44}\right) \\ \frac{k^2}{2}\left(C_{12}+C_{44}\right) & \frac{k^2}{2}\left(C_{11}+C_{44}\right)-\rho\omega^2 \end{pmatrix}\begin{pmatrix} u_x \\ u_y \end{pmatrix} = 0$$

To have non zero solns, $\det(\;) = 0$

$$\left[\frac{k^2}{2}\left(C_{11}+C_{44}\right)-\rho\omega^2\right]^2 - \left[\frac{k^2}{2}\left(C_{12}+C_{44}\right)\right]^2 = 0$$

$$\frac{k^2}{2}\left(C_{11}+C_{44}\right)-\rho\omega^2 = \pm\frac{k^2}{2}\left(C_{12}+C_{44}\right)$$

$$\frac{\omega^2}{k^2} = \frac{1}{2\rho}\left[\left(C_{11}+C_{44}\right) \pm \left(C_{12}+C_{44}\right)\right]$$

Two solns, either

$$\boxed{\frac{\omega}{k} = \sqrt{\frac{1}{2\rho}\left(C_{11}+C_{12}+2C_{44}\right)}}$$

or

$$\boxed{\frac{\omega}{k} = \sqrt{\frac{1}{2\rho}\left(C_{11}-C_{12}\right)}}$$

To determine direction of oscillation for these velocities, plug that $\frac{\omega}{k}$ value back into Eqn 3.57 and solve for $\frac{u_y}{u_x}$ ratio. One turns out to be $u_y = u_x$, the other $u_y = -u_x$.