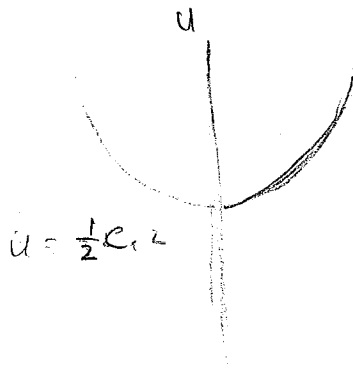


spring:



Schrodinger:  $U(x) \psi(x) - \frac{\hbar^2}{2m} \frac{d^2 \psi(x)}{dx^2} = E_n \psi(x)$

often notation  $(U - \frac{\hbar^2}{2m} \frac{d^2}{dx^2}) \psi = E \psi$

$\frac{1}{2} kx^2 \psi - \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi$

let  $\omega = \sqrt{\frac{k}{m}}$

$\frac{1}{2} m \omega^2 x^2 \psi - \frac{\hbar^2}{2m} \frac{d^2 \psi}{dx^2} = E \psi$

com, ma

$\frac{d^2 \psi}{dx^2} = -k \psi$

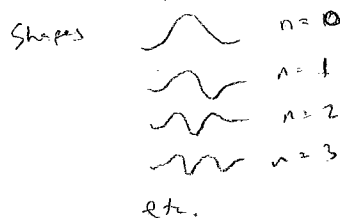
Solns  $\psi = A \cos kx$   
 $\psi = B \sin kx$

linear comb.

Solns: linear comb of infinite # of "eigenfunctions"

Each eigenfunction yields a different E

$\psi_n$  = complicated formula involving "Hermite polynomials"



or  $\frac{d^2 \psi}{dx^2} = +k \psi$

Solns  $\psi = A e^{kx}$   
 or  $\psi = B e^{-kx}$

$E_n$  = simple formula!

$E_n = \hbar \omega (n + \frac{1}{2})$

Operators

consider  $a = \sqrt{\frac{m\omega}{2\hbar}} \left( x + \frac{\hbar}{m\omega} \frac{d}{dx} \right)$

$a^\dagger = \sqrt{\frac{m\omega}{2\hbar}} \left( x - \frac{\hbar}{m\omega} \frac{d}{dx} \right)$

called operators because they only tell you what to do w/  $\psi$ .  
Don't have independent meaning

What is  $\hbar\omega a^\dagger a \psi$ ? (be w/ me...)

$= \hbar\omega \frac{m\omega}{2\hbar} \left( x - \frac{\hbar}{m\omega} \frac{d}{dx} \right) \left( x + \frac{\hbar}{m\omega} \frac{d}{dx} \right) \psi$

Not difference of squares, because operators don't work like that  
use FOIL

$= \frac{m\omega^2}{2} \left( x^2 \psi + \cancel{\frac{\hbar}{m\omega} x \frac{d\psi}{dx}} - \frac{\hbar}{m\omega} \frac{d}{dx} (x\psi) - \frac{\hbar^2}{m^2\omega^2} \frac{d^2\psi}{dx^2} \right)$

↓  
 $x \frac{d\psi}{dx} + \psi \frac{d}{dx} x$   
 $\cancel{x \frac{d\psi}{dx}} + \psi$

$= \frac{m\omega^2}{2} \left( x^2 \psi - \frac{\hbar}{m\omega} \psi - \frac{\hbar^2}{m^2\omega^2} \frac{d^2\psi}{dx^2} \right)$

$\hbar\omega a^\dagger a \psi = \frac{m\omega^2}{2} x^2 \psi - \frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} - \frac{1}{2} \hbar\omega \psi$

$(a^\dagger a + \frac{1}{2}) \hbar\omega \psi =$  = LHS of Schrodinger Eqn!

Schro. Eqn:  $(a^\dagger a + \frac{1}{2}) \hbar\omega \psi_n = (n + \frac{1}{2}) \hbar\omega \psi_n$

$a^\dagger a \psi_n = n \psi_n$

$a^\dagger a \leftrightarrow n$

called the "number operator"

not the same, but related!  
(like eigenvector / eigenvalue  
 $Mv = \lambda v$ )

Similarly  $(a^\dagger a + \frac{1}{2}) \hbar\omega$  called "energy operator" or "Hamiltonian" gives symbol H

(looking back at Schrod. Eqn)  $-\frac{\hbar^2}{2m} \frac{d^2}{dx^2}$  called KE operator

compare KE =  $\frac{p^2}{2m}$

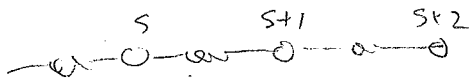
$\rightarrow -\hbar^2 \frac{d^2}{dx^2}$

called momentum operator symbol  $\hat{p}$  or often just  $p$   
(I guess negative sign not obvious)

Schrodinger Eqn  $\frac{1}{2} C x^2 \psi + \frac{p^2}{2m} \psi = E \psi$

but not really momentum

Appendix 2



$m = \text{atoms}$   
"w/aperiodic" body

modification:  $U = \text{potential energy of all springs} = \sum_{s=1}^N \frac{1}{2} c (u_{s+1} - u_s)^2$

↳ Hamiltonian  $H$ , instead of  $U_{tot}$

$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi}{\partial x^2} \rightarrow$  term for each atom

$\frac{p_x^2}{2m}$  using operator notation

Change of variables

$Q_k =$  discrete FT of  $u_s$

$P_k =$  discrete IFT of  $p_s$

(depending on details of FT + IFT, which one often backward for discrete case)

↑ summation over  $s$  instead of integral over  $x$ .  
Some other differences

$k =$  discrete wavevector, spacing  $\frac{1}{N} \times \frac{2\pi}{a}$  as perhaps expected  
from  $-\frac{\pi}{a}$  to  $+\frac{\pi}{a}$

Result of transform is

$$H = \sum_k \left[ \frac{1}{2} m \left( \frac{2c}{m} (1 - \cos ka) \right) |Q_k|^2 + \frac{1}{2m} |P_k|^2 \right]$$

where  $P$  and  $Q$  have same relationship to each other that  $p$  and  $x$  do.  
(or  $u$ )

↳ I'm not positive I got this right. Book lists  $P_k P_{-k}$ , I think  $P_{-k} = P_k^*$  but couldn't prove it w/ Eqn is book.

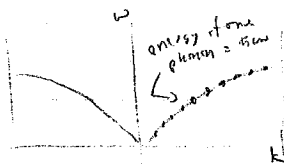
Translation: For a given  $k$ , system acts like

harmonic oscillator having  $\omega_k = \sqrt{\frac{2c}{m} (1 - \cos ka)}$

← our previous dispersion relation!

Allowed Energies at that  $k$  are  $E_{k,n} = (n + \frac{1}{2}) \hbar \omega_k$

$n =$  number of something, call it # phonons.



Just plot regular  $\omega$  vs  $k$

but consider that can get multiple phonons for a given  $k$ .

(and usually ignore zero pt energy)

Back to Chapter 4

Book analysis  $u_0 = \sqrt{\frac{4(n + \frac{1}{2}) \hbar}{\rho V \omega}}$   
↳ vol. of crystal

= rms displacement for a given mode, with  $n$  given # phonons

compare to photons:

photons: - light comes in bundles of energy,  $E = hf$  or  $= \hbar\omega$

- each additional bundle will increase the net electric field according to field amplitude  $\propto \sqrt{n}$   
(since Energy  $\propto$  amplit.<sup>2</sup>)

- each bundle has no "mass" (rest energy)

I.e. for very low  $\omega$  photons, "cost" to create  $\rightarrow 0$

Just the same!

- wavelength  $\lambda = \frac{c}{f} = \frac{hc}{E}$

- momentum  $p = \frac{E/c}{c} = \frac{h}{\lambda} = \frac{h}{(2\pi/k)} = \hbar \cdot k$

Direct analogy:

photon momentum  $\vec{p} = \hbar \vec{k}$

"crystal momentum"

One difference: remember  $\vec{k}$  is only defined modulo a  $2\pi/a$ .

So this behaves a little differently than you might expect.

But aside from that, we have

cons. of momentum as usual