Next Goal: calculate heat capacity of electron gas

$$U = \int_0^\infty \underbrace{D(E)}_{\text{density of states}} \; \underbrace{f(E)}_{\text{probability of state being occupied}} \; \underbrace{E}_{\text{energy of state}} \; dE$$

{ Same as for phonons except we'll use E instead of ω }

then $C_v = \frac{\partial U}{\partial T}$ Compare: $Q = \underbrace{n C_v}_{\text{this is what we're calling } C_v} \Delta T$ (for constant volume, no work, so $\Delta E = Q + \cancel{W}^{0}$)

Step 1: understand $f(E)$ a little better
→ Fermi Dirac distribution

Step 2: calculate $D(E)$ for electrons
(review how we did it for eg. acoustic phonons)

Step 3. piece together

Fermi - Dirac Distribution in detail

$f = n_{ave} = \dfrac{1}{e^{(E-\mu)/kT}+1}$

how many particles at energy E

μ = chemical potential, energy needed to add a particle

= 0 for photons but

- important for electrons, depends on T

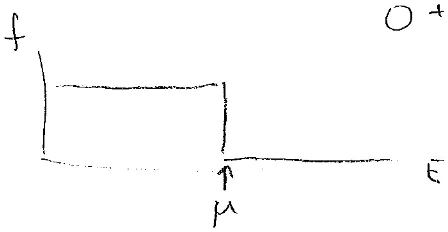
Also: μ is point where $f = \frac{1}{2}$. When $E = \mu$, $f = \frac{1}{2}$ (see from eqn)

$\underline{\underline{T = 0}}$: $f = \dfrac{1}{e^{(E-\mu)/kT}+1}$ = (if exponent positive, $E > \mu$)

$\dfrac{1}{\text{huge} + 1} = 0$

= (if exponent is negative, $E < \mu$)

$\dfrac{1}{0+1} = 1$



empty [

full [

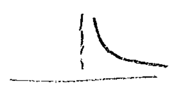
$\mu(T=0)$ called "Fermi Energy", E_F

Either a state is occupied, or it's not

$T = \underline{\underline{\text{small}}}$ but not zero

if $E > \mu$, $\dfrac{1}{\text{huge}+1} = \dfrac{1}{\text{huge}} = \dfrac{1}{e^{(E-\mu)/kT}} = e^{-(E-\mu)/kT}$

$= e^{-x/kT}$



$x < 0$
$E = \mu$

if $E < \mu$ $\dfrac{1}{\text{small}+1} = (1+\text{small})^{-1}$

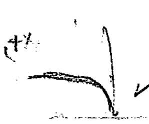
$= 1 - \text{small}$

$= 1 - e^{+x/kT}$

?

$1 - e^{-x/kT}$

$x < 0$



at $E = \mu$ predicts 1

At $E = \mu$ predicts 0

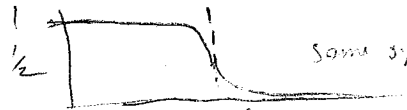
Actually: $f = \frac{1}{2}$ $\rightarrow \mu = E_F$

all empty [

some prob. of being full [

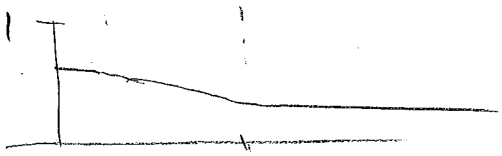
all full [

put together



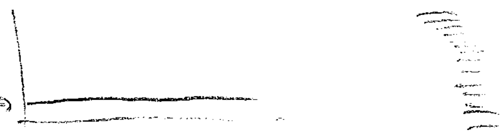
some symmetry... total area the same

T = high



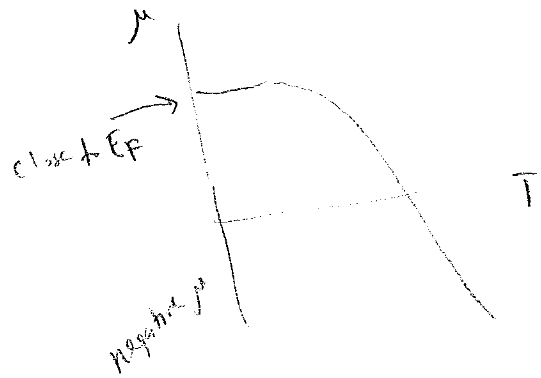
very long tail
drags low E state probability down

T = extremely high



Note: when $(E-\mu) \gg k_B T$ $\longrightarrow$ small values of f
$\longrightarrow$ get Boltzmann statistics again

μ = point where $f = \frac{1}{2}$



Solve for μ via $N = \int_0^{\infty} D(E) f(E)\, dE$

for low T we'll say $\mu \simeq E_F$

For more accurate $\rightarrow$ see HW problem

DOS review – what we did for phonons

$$D(\omega) = \frac{1}{\left(\frac{2\pi}{V}\right)^3} \times 4\pi k^2 \times \left|\frac{1}{\frac{d\omega}{dk}}\right|$$

pts / k-space volume

Shell model, spherical approx.

slope of dispersion

Could have multiplied by 3 (for 3 acoustic branches), depending on defn of $D(\omega)$. We waited until U calculation to do ×3

Debye
Acoustic phonons: $\omega = vk$ approximation $\rightarrow k = \frac{\omega}{v}$

$d\omega = v\,dk \rightarrow \frac{d\omega}{dk} = v$

$$D(\omega) = \frac{Vol}{8\pi^3} \cdot 4\pi\left(\frac{\omega}{v}\right)^2 \frac{1}{v}$$

Electrons: want an integral of E, not ω

$$D(\epsilon) = 2 \times \frac{1}{\left(\frac{2\pi}{L}\right)^3} \times 4\pi k^2 \frac{1}{\frac{dE}{dk}}$$

\# spin states (included here, by convention)

take on faith for now

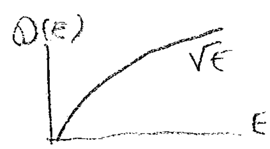
one handout

$E = \frac{\hbar^2 k^2}{2m} \longrightarrow k^2 = \frac{2mE}{\hbar^2}$

$dE = \frac{\hbar^2 k}{m} dk \longrightarrow \frac{dE}{dk} = \frac{\hbar^2 k}{m} = \frac{\hbar^2}{m}\left(\sqrt{\frac{2mE}{\hbar^2}}\right)$

$$D(E) = 2\frac{V}{(2\pi)^3} 4\pi \left(\frac{2mE}{\hbar^2}\right) \frac{1}{\frac{\hbar^2}{m}\sqrt{\frac{2mE}{\hbar^2}}}$$

$$\boxed{D(E) = \frac{Vol}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}}$$

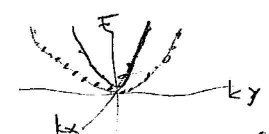


proof of factors of 2 in last step

can skip?

$$\frac{2 \times 4 \times 2}{\sqrt{2} \times 2^3} = \frac{2}{\sqrt{2}} = \sqrt{2}$$

also $\frac{2^{3/2}}{2} = \sqrt{2}$

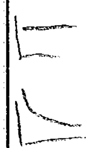


2D can sort of visualize

3D hard

1D and 2D for HW ✓

1D on ^? 2D for any n?



→ 1D D(E) diverges (at origin?)

Review: $U = \int E \, \mathcal{D}(E) \frac{1}{e^{(E-\mu)/kT}+1} \, dE$

from $\mathcal{D}(E) = 2 \times \frac{1}{\left(\frac{2\pi}{L}\right)^3} \, 4\pi k^2 \, \frac{1}{\frac{dE}{dk}}$

that was step (a)

$\mathcal{D}(E) = \frac{V}{2\pi^2} \left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}$

Want to get $C = \frac{\partial U}{\partial T}$

Need to find μ.

handout

A

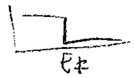
We'll say for low T, $\mu \simeq E_F$ (remember, we proved this w/ Taylor series)

How to find E_F? It's $\mu(T=0)$

Two methods - see next page

Calculating the Fermi Energy

$$N = \int_0^\infty D(E)\, f(E)\, dE$$

at 0K $f(E) =$ 

$$= \int_0^{E_F} D(E)\, dE$$

$$= \int_0^{E_F} \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \sqrt{E}\, dE$$

$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \left.\frac{E^{3/2}}{3/2}\right|_0^{E_F}$$

$$N = \frac{V}{3\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E_F^{3/2}$$

$$E_F = \left[3\pi^2 \frac{N}{V}\left(\frac{\hbar^2}{2m}\right)^{3/2}\right]^{2/3}$$

$$\boxed{E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}}$$

Comments on 1D + 2D

Easier way?: do it in k-space

$$E = \frac{p^2}{2m} = \frac{\hbar^2 k^2}{2m}$$

$$E_F = \frac{\hbar^2 k_F^2}{2m}$$

→ volume × $\frac{\text{pts}}{\text{vol}}$ = # pts

× 2 (spin states)

$$N = \frac{4}{3}\pi k_F^3 \cdot \frac{1}{(2\pi/L)^3} \cdot 2 = \frac{4}{3}\pi \frac{V}{8\pi^3} 2\, k_F^3$$

$$= \frac{1}{3\pi^2} V k_F^3$$

$$k_F = \left(3\pi^2 \frac{N}{V}\right)^{1/3}$$

$$\boxed{E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}}$$ same!

end of step (b)