Heat Capacity of 3D Electron Gas – Summary by Dr Colton

(a) Density of states

$$D(E) = \frac{1}{(2\pi/L)^3} (4\pi k^2) \frac{1}{dE/dk} \times \underbrace{2}_{\text{spin states}}$$

$$\text{use } E = \frac{\hbar^2 k^2}{2m} \rightarrow k = \frac{1}{\hbar}\sqrt{2mE}$$

$$\frac{dE}{dk} = \frac{\hbar^2 k}{m} = \frac{\hbar^2}{m}\left(\frac{1}{\hbar}\sqrt{2mE}\right)$$

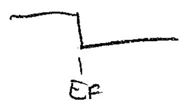
$$= \hbar\sqrt{\frac{2E}{m}}$$

$$D(E) = \frac{V}{8\pi^3}\left[4\pi\left(\frac{2mE}{\hbar^2}\right)\right]\left[\frac{1}{\hbar\sqrt{\frac{2E}{m}}}\right](2)$$

$$\boxed{D(E) = \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} E^{1/2}} \qquad (1)$$

(b) Fermi Energy

$$N = \int_0^\infty D(E) f(E)\, dE$$

$f(E) =$  at 0 deg kelvin

$$N = \int_0^{E_F} D(E)\, dE$$

$$= \frac{V}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2} \underbrace{\int_0^{E_F} E^{1/2}\, dE}_{\frac{E_F^{3/2}}{3/2}}$$

$$\boxed{E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}} \qquad (2)$$

Alternate derivation:  # states = volume × $\frac{\text{\# states}}{\text{volume}}$ × 2

$$N = \left(\frac{4}{3}\pi k_F^3\right)\frac{1}{(2\pi/L)^3} \times 2$$

$$N = \frac{V}{3\pi^2} k_F^3$$

$$= \frac{V}{3\pi^2}\left(\frac{1}{\hbar}\sqrt{2mE_F}\right)^3$$

$$\boxed{E_F = \frac{\hbar^2}{2m}\left(3\pi^2 \frac{N}{V}\right)^{2/3}} \checkmark$$

(c) Calculate $\mathcal{D}(\epsilon_F)$ ... Solve Eqn (2) for V and plug into Eqn (1)

$$\left(\frac{2m\epsilon_F}{\hbar^2}\right)^{3/2} = \frac{3\pi^2 N}{V}$$

$$V = 3\pi^2 N\left(\frac{\hbar^2}{2m\epsilon_F}\right)^{3/2}$$

then $\mathcal{D}(\epsilon_F) = \left[3\pi^2 N\left(\frac{\hbar^2}{2m\epsilon_F}\right)^{3/2}\right]\frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\epsilon_F^{1/2}$

$$\boxed{\mathcal{D}(\epsilon_F) = \frac{3}{2}\frac{N}{\epsilon_F}}$$

(d) Energy: $U = \int_0^\infty \underbrace{\mathcal{D}(\epsilon)}_{\text{approximate this as } \mathcal{D}(\epsilon_F)} \, f(\epsilon)\, \epsilon \, d\epsilon$

$f(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/kT}+1}$ approximate μ as ϵ_F, a constant

$$\boxed{U \approx \mathcal{D}(\epsilon_F)\int_0^\infty \frac{\epsilon \, d\epsilon}{e^{(\epsilon-\epsilon_F)/kT}+1}}$$

(e) heat capacity: $C_V = \frac{\partial U}{\partial T}$ → take $\frac{\partial}{\partial T}$ inside the integral

$$\frac{\partial}{\partial T}\left(\frac{1}{e^{(\epsilon-\epsilon_F)/kT}+1}\right) = \frac{-1}{\left(e^{(\epsilon-\epsilon_F)/kT}+1\right)^2}\left(e^{(\epsilon-\epsilon_F)/kT}\right)\left(\frac{\epsilon-\epsilon_F}{k}\cdot\frac{-1}{T^2}\right)$$

$$C_V = \mathcal{D}(\epsilon_F)\frac{1}{kT^2}\int_0^\infty \frac{(\epsilon-\epsilon_F)\,\epsilon\, e^{(\epsilon-\epsilon_F)/kT}}{\left(e^{(\epsilon-\epsilon_F)/kT}+1\right)^2}d\epsilon$$

let $x = \frac{\epsilon-\epsilon_F}{kT}$ → $\epsilon = kTx + \epsilon_F$

$d\epsilon = kT\,dx$

$$\boxed{C_V = \mathcal{D}(\epsilon_F)\, k\int_{-\epsilon_F/kT}^{\infty}\frac{x e^x}{(e^x+1)^2}(kTx+\epsilon_F)\,dx}$$

(f) small T approx: take integral from $-\infty$ to ∞, then — this part integrates to 0 since $\frac{xe^x}{(e^x+1)^2}$ = odd

$$C_V = \mathcal{D}(\epsilon_F)\,k^2 T\underbrace{\int_{-\infty}^{\infty}\frac{x^2 e^x}{(e^x+1)^2}dx}_{\pi^2/3}$$

$$C_V = \left(\frac{3}{2}\frac{N}{\epsilon_F}\right)(k^2T)\left(\frac{\pi^2}{3}\right)$$

$$\boxed{C_V = \frac{\pi^2}{2}\frac{k^2T}{\epsilon_F}N}$$

the contribution to the heat capacity from the electron gas.