Assignment 10

Arfken 5.10.1

Show that Stirling's formula is an asymptotic expansion. The remainder term is

$$R_N(x) = -\sum_{n=N+1} \frac{B_{2n}}{2n(2n-1)} x^{1-2n}$$

for some $N \geq 1$. The condition for an asymptotic series,

$$\lim_{x\to\infty} x^N R_N = -\lim_{x\to\infty} \sum_{n=N+1} \frac{B_{2n}}{2n(2n-1)} \frac{1}{x^{2n-1-N}} = 0$$

is thus met. We should also check that the series formally diverges. We can do that using $\lim_{N\to\infty} x^N R_N(x)$ or just use the ratio test on the series (and using the representation of the Bernoulli numbers given in equation 5.152)

$$\begin{aligned}
\left|\frac{a_{n+1}}{a_n}\right| &= \frac{B_{2n+2}\, x^{-1-2n}}{(2n+2)(2n+1)} \frac{2n(2n-1)}{B_{2n}\, x^{1-2n}} \\
&= \frac{(2n+2)!\,(2\pi)^{2n}\,\zeta(2n+2)}{(2n)!\,(2\pi)^{2n+2}\,\zeta(2n)} \frac{2n(2n-1)}{(2n+2)(2n+1)} \frac{1}{x^2} \\
&= \frac{1}{(2\pi)^2} \frac{\zeta(2n+2)}{\zeta(2n)} 2n(2n-1) \frac{1}{x^2}
\end{aligned}$$

which obviously $\to \infty$ as $n \to \infty$ (note that $\lim_{n\to\infty} \zeta(n) = 1$). Thus the Stirling series is an asymptotic expansion.

Arfken 5.10.2

Let's do both Fresnel integrals together. But first note that the infinite version of both of these integrals is $\int_0^\infty \cos \pi u^2/2 \, du = 1/2$ (getting this from a table). So use this to define an asymptotic series

$$\begin{aligned}
C(x) + iS(x) &= \int_0^x \cos\frac{\pi u^2}{2} + i\int_0^x \sin\frac{\pi u^2}{2}\, du \\
&= \int_0^\infty e^{i\pi u^2/2}\, du - \int_x^\infty e^{i\pi u^2/2}\, du \\
&= \frac{1}{2}(1+i) - \frac{1}{\sqrt{2\pi}} \int_{\pi x^2/2}^\infty \frac{e^{iz}}{\sqrt{z}}\, dz \\
&= \frac{1}{2}(1+i) - \frac{1}{\sqrt{2\pi}} \left[\frac{e^{iz}}{iz^{1/2}} + \frac{1}{2i}\int \frac{e^{iz}}{z^{3/2}}\, dz\right]\Bigg|_{\pi x^2/2}^\infty \\
&= \frac{1}{2}(1+i) - \frac{1}{\sqrt{2\pi}} \left[\frac{e^{iz}}{iz^{1/2}} + \frac{1}{2i^2}\frac{e^{iz}}{z^{3/2}} + \frac{3}{4i^2}\int \frac{e^{iz}}{z^{5/2}}\, dz\right]\Bigg|_{\pi x^2/2}^\infty \\
&\sim \frac{1}{2}(1+i) + \frac{1}{\pi x}\left[\left(-i\cos\frac{\pi x^2}{2} + \sin\frac{\pi x^2}{2}\right) - \frac{1}{2\pi x^2/2}\left(\cos\frac{\pi x^2}{2} + i\sin\frac{\pi x^2}{2}\right)\right. \\
&\qquad \left. - \frac{3}{4\pi^2 x^4/4}\left(-i\cos\frac{\pi x^2}{2} + \sin\frac{\pi x^2}{2}\right) + \frac{15}{8\pi^3 x^6/8}\left(\cos\frac{\pi x^2}{2} + i\sin\frac{\pi x^2}{2}\right) + \cdots\right] \\
&\sim \frac{1}{2} + \frac{1}{\pi x}\left[\sin\frac{\pi x^2}{2} - \frac{1}{\pi x^2}\cos\frac{\pi x^2}{2} - \frac{3}{\pi^2 x^4}\sin\frac{\pi x^2}{2} + \frac{15}{\pi^3 x^6}\cos\frac{\pi x^2}{2} + \cdots\right] \\
&\quad + \frac{i}{2} + \frac{i}{\pi x}\left[-\cos\frac{\pi x^2}{2} - \frac{1}{\pi x^2}\sin\frac{\pi x^2}{2} + \frac{3}{\pi^2 x^4}\cos\frac{\pi x^2}{2} + \frac{15}{\pi^3 x^6}\sin\frac{\pi x^2}{2} + \cdots\right]
\end{aligned}$$

Thus the real part of this is $C(x)$ and the imaginary part is $S(x)$.

Arfken 5.10.5

For the series

$$P_\nu(z) \sim 1 + \sum_{n=1}^{\infty} (-1)^n \frac{\prod_{s=1}^{2n} \left[4\nu^2 - (2s-1)^2\right]}{(2n)!\,(8z)^{2n}}$$

$$Q_\nu(z) \sim \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\prod_{s=1}^{2n-1} \left[4\nu^2 - (2s-1)^2\right]}{(2n-1)!\,(8z)^{2n-1}}$$

the remainder terms are, respectively,

$$R_n^P(z) = \sum_{k=n+1}^{\infty} (-1)^k \frac{\prod_{s=1}^{2k} \left[4\nu^2 - (2s-1)^2\right]}{(2k)!\,(8z)^{2k}}$$

$$R_n^Q(z) = \sum_{k=n+1}^{\infty} (-1)^{k+1} \frac{\prod_{s=1}^{2k-1} \left[4\nu^2 - (2s-1)^2\right]}{(2k-1)!\,(8z)^{2k-1}}$$

and both quantities, $z^n R_n(z)$, approach zero as $z \to \infty$ since the z terms go like z^{-2k+n} and $z^{-2k+1+n}$ and $k > n$.

To demonstrate that both are formally divergent series, use the ratio test for $P_\nu(z)$

$$\begin{aligned}\left|\frac{a_{n+1}}{a_n}\right| &= \frac{\prod_{s=1}^{2n+2} \left[4\nu^2 - (2s-1)^2\right]}{(2n+2)!\,(8z)^{2n+2}} \frac{(2n)!\,(8z)^{2n}}{\prod_{s=1}^{2n} \left[4\nu^2 - (2s-1)^2\right]} \\ &= \frac{\left(4\nu^2 - (4n-3)^2\right)\left(4\nu^2 - (4n-1)^2\right)}{(2n+1)(2n+2)(8z)^2}\end{aligned}$$

which, for a fixed z, goes to ∞ as $n \to \infty$. A similar calculation follows for $Q_\nu(z)$.

Arfken 5.10.8

We want to expand the integral

$$\begin{aligned}\int_0^\infty e^{-xv}(1+v^2)^{-2}\,dv &= \int_0^\infty e^{-u}\left(1+\frac{u^2}{x^2}\right)^{-2} \frac{1}{x}\,du \qquad \text{where} \quad u = xv \\ &= \frac{1}{x}\int_0^\infty e^{-u} \sum_{k=0}^{\infty} (-1)^k (k+1) \left(\frac{u^2}{x^2}\right)^k du \\ &\sim \frac{1}{x}\sum_{k=0}^{n} (-1)^k (k+1) \frac{1}{x^{2k}} \int_0^\infty e^{-u} u^{2k} du \\ &\sim \sum_{k=0}^{n} (-1)^k (k+1) \frac{(2k)!}{x^{2k+1}}\end{aligned}$$

Note that the book would seem to have an error. My guess is that the -2 in the exponent of the integral was really supposed to be -1.

Arfken 3.1.1

(a)

$$\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{vmatrix} = -1$$

(b)

$$\begin{vmatrix} 1 & 2 & 0 \\ 3 & 1 & 2 \\ 0 & 3 & 1 \end{vmatrix} = 1 \cdot (1 \cdot 1 - 3 \cdot 2) - 2 \cdot (3 \cdot 1 - 2 \cdot 0) + 0 \cdot (3 \cdot 3 - 0 \cdot 1) = -11$$

(c)

$$\begin{aligned} \frac{1}{\sqrt{2}} \begin{vmatrix} 0 & \sqrt{3} & 0 & 0 \\ \sqrt{3} & 0 & 2 & 0 \\ 0 & 2 & 0 & \sqrt{3} \\ 0 & 0 & \sqrt{3} & 0 \end{vmatrix} &= -\sqrt{\frac{3}{2}} \begin{vmatrix} \sqrt{3} & 2 & 0 \\ 0 & 0 & \sqrt{3} \\ 0 & \sqrt{3} & 0 \end{vmatrix} \\ &= -\sqrt{\frac{3}{2}} (\sqrt{3})^3 \\ &= -\frac{9\sqrt{2}}{2} \end{aligned}$$

Arfken 3.1.6a

$$\begin{aligned} D_1 &= |\,1\,| = 1 \\ D_2 &= \begin{vmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{vmatrix} \\ &= \frac{1}{3} - \frac{1}{4} \\ &= \frac{1}{12} \\ D_3 &= \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{3} & \frac{1}{4} \\ \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \end{vmatrix} \\ &= 1 \cdot \left(\frac{1}{15} - \frac{1}{16} \right) - \frac{1}{2} \cdot \left(\frac{1}{10} - \frac{1}{12} \right) + \frac{1}{3} \left(\frac{1}{8} - \frac{1}{9} \right) \\ &= \frac{1}{2160} \end{aligned}$$

Arfken 3.2.4

(a) If complex numbers can be represented by 2×2 matrices, then we should be able to reproduce the basic arithmetic operations in a consistent manner. For instance addition of two complex numbers: $(a+ib)+(c+id)$ becomes

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} + \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} a+c & b+d \\ -(b+d) & a+c \end{pmatrix}$$

which, translated back to complex numbers, is $(a+c)+i(b+d)$ as we would expect for an isomorphic representation.

Likewise, multiplication of two complex numbers $(a+ib) \cdot (c+id)$ becomes

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix} \cdot \begin{pmatrix} c & d \\ -d & c \end{pmatrix} = \begin{pmatrix} ac - bd & bd + ac \\ -(bc + ad) & -bd + ac \end{pmatrix}$$

which, translated back to complex numbers, is $(ac - bd) + i(bd + ac)$ and which is complex mutiplication. Thus the algebra of complex numbers is isomorphic to the algebra of matrices which have the form

$$\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$$

(b) The matrix corresponding to $(a + ib)^{-1}$ is just the inverse of the standard matrix. Finding the inverse, we get

$$\frac{1}{a^2 + b^2} \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

which in complex notation is $\frac{a-ib}{a^2+b^2}$ as expected.

Arfken 3.2.6a

Starting with a general 2×2 matrix

$$A = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

the demand that $A \cdot A = 0$ leads to four equations:

$$\begin{aligned} x^2 &= -yz \\ y(x + w) &= 0 \\ z(x + w) &= 0 \\ w^2 &= -yz \end{aligned}$$

One solution is the trivial solution: $x = y = z = w = 0$ which we discard. The other solution is $w = -x$ and $y = -x^2/z$ with x and z arbitrary. The matrix becomes

$$A = \begin{pmatrix} x & -x^2/z \\ z & -x \end{pmatrix}$$

and if we make the redefinitions $x = ab$ and $z = -a^2$, we get the form in the text.

Arfken 3.2.14

From the previous problem, we have the anti-commutation relations and can deduce the commutation relations between the Pauli matrices:

$$\begin{aligned} \sigma_i \, \sigma_j + \sigma_j \, \sigma_i &= 2\delta_{ij} \, \mathbf{1} \\ \sigma_i \, \sigma_j - \sigma_j \, \sigma_i &= 2i\epsilon_{ijk} \, \sigma_k \end{aligned}$$

Adding these equations and dividing by 2 we get

$$\sigma_i \, \sigma_j = \delta_{ij} \, \mathbf{1} + i\epsilon_{ijk} \, \sigma_k$$

Since $\vec{\sigma}$ is a *matrix*-valued vector (this is the same language as when we speak of a real-valued function) we can dot an “ordinary” vector (*i.e.* a real-valued vector) into it

$$(\sigma_i a_i) \; (\sigma_j a_j) = a_i b_j \delta_{ij} \, \mathbf{1} + i\epsilon_{kij} \, \sigma_k a_i b_j$$

or in traditional vector notation,

$$(\vec{\sigma} \cdot \vec{a}) \left(\vec{\sigma} \cdot \vec{b}\right) = \vec{a} \cdot \vec{b} \, \mathbf{1} + i\vec{\sigma} \cdot \left(\vec{a} \times \vec{b}\right)$$

Arfken 3.2.16

(a)

$$
\begin{aligned}
[M_x, M_y] &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= i \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= iM_z
\end{aligned}
$$

with similar calculations for $[M_y, M_z] = iM_x$ and $[M_z, M_x] = iM_y$.

(b)

$$
\begin{aligned}
M^2 &\equiv {M_x}^2 + {M_y}^2 + {M_z}^2 \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= 2\,\mathbf{1}
\end{aligned}
$$

(c) Using just the commutation relations we get

$$
\begin{aligned}
[M^2, M_i] &= [2\,\mathbf{1}, M_i] \\
&= 2\,[\mathbf{1}, M_i] \\
&= 0
\end{aligned}
$$

since the identity matrix commutes with everything.

$$
\begin{aligned}
[M_z, L^+] &= [M_z, M_x + iM_y] \\
&= [M_z, M_x] + i[M_z, M_y] \\
&= iM_y + i\,(-iM_x) \\
&= L^+
\end{aligned}
$$

Lastly,

$$
\begin{aligned}
[L^+, L^-] &= [M_x + iM_y, M_x - iM_y] \\
&= [M_x, M_x] - i[M_x, M_y] + i[M_y, M_x] + [M_y, M_y] \\
&= 0 - i\,(iM_z) + i\,(-iM_z) + 0 \\
&= 2M_z
\end{aligned}
$$

Arfken 3.2.23 The matrix A is diagonal and commutes with the matrix B. Show B is diagonal.

The commutator in index notation is

$$\begin{aligned} 0 &= A_{ij} B_{jk} - B_{ij} A_{jk} \\ &= A_{ii'} B_{i'k} - B_{ik'} A_{k'k} \end{aligned}$$

where our notation is a bit strange here. We are summing over i' and k', however, since A is diagonal, there is only a *single* term in each of these sums. The notation is to emphasize the fact that $i' = i$ and $k' = k$, again, because A is diagonal. Therefore, since $i' = i$ and $k' = k$, $B_{i'k} = B_{ik'}$ and we can write

$$0 = B_{i'k}(A_{ii'} - A_{k'k})$$

which, if the diagonal elements of A are distinct (assumed) implies that what is in parentheses is nonzero for $i \neq k$ and hence $B_{i'k} = 0$ for $i' \neq k$ and B is diagonal.

Arfken 3.2.28

The matrices A and B satisfy $A^2 = B^2 = \mathbf{1}$ and

$$\{A, B\} \equiv AB + BA = 0$$

Multiply the above anti-commutation relation by A and take the trace:

$$\begin{aligned} 0 &= \mathrm{tr}\,(AAB + ABA) \\ &= \mathrm{tr}\,(B + AAB) \\ &= 2\,\mathrm{tr}\,(B) \end{aligned}$$

where we have used $A^2 = \mathbf{1}$ and the cyclic property of the trace.

A virtually identical calculation shows the same thing for $\mathrm{tr}\,(A)$.

Arfken 3.3.1

The matrices A and B are orthogonal: $A^T = A^{-1}$ and $B^T = B^{-1}$. Consider the transpose of their product in index notation

$$\begin{aligned} \left((A\,B)^T\right)_{ij} &= (A_{ik} B_{kj})^T \\ &= A_{jk} B_{ki} \\ &= B_{ki} A_{jk} \\ &= (B_{ik})^T\,(A_{kj})^T \\ &= \left(B^T A^T\right)_{ij} \end{aligned}$$

Thus we can now write

$$\begin{aligned} (A\,B)^T &= B^T A^T \\ &= B^{-1} A^{-1} \end{aligned}$$

If we multiply this by AB from the left, we get the identity, $\mathbf{1}$ which establishes $(A\,B)^T$ as the inverse of AB and hence AB as an orthogonal matrix.

Arfken 3.3.8

In index notation, we can write our symmetric matrix as $(S)_{ij} = s_{ij} = s_{ji}$ and the anti-symmetric matrix as $(A)_{ij} = a_{ij} = -a_{ji}$. The trace is then

$$\begin{aligned} \mathrm{tr}\,(S\,A) &= s_{ij}\,a_{ji} \\ &= (s_{ji})\,(-a_{ij}) \\ &= -s_{ji} a_{ij} \\ &= -\mathrm{tr}\,(S\,A) \end{aligned}$$

and we have the trace equal to its negative, thus it is zero.

Arfken 3.3.9

For a similarity transformation, we have $A' = B\,A\,B^{-1}$. Taking the trace of this, we have

$$\begin{aligned} \mathrm{tr} A' &= \mathrm{tr}\left(B\,A\,B^{-1}\right) \\ &= \mathrm{tr}\left(B^{-1}\,B\,A\right) \\ &= \mathrm{tr}\,(A) \end{aligned}$$

and the trace of a matrix is invariant under similarity transformations.

Arfken 3.3.10

For a similarity transformation, we have $A' = B\,A\,B^{-1}$. Taking the determinant of this, we have

$$\begin{aligned} \det A' &= \det\left(B\,A\,B^{-1}\right) \\ &= (\det\ B)\ (\det\ A)\ \left(\det\ B^{-1}\right) \\ &= (\det\ B)\ (\det\ A)\ (\det\ B)^{-1} \\ &= \det\ A \end{aligned}$$

and the determinant of a matrix is invariant under similarity transformations.