Assignment 11

Arfken 3.4.6

Matrix C is not Hermition. But

$$\begin{aligned}(C + C^\dagger)^\dagger &= (C^\dagger + C^{\dagger\dagger}) \\ &= (C^\dagger + C)\end{aligned}$$

which is Hermitian. Likewise,

$$\begin{aligned}\left[i(C - C^\dagger)\right]^\dagger &= -i(C^\dagger - C^{\dagger\dagger}) \\ &= -i(C^\dagger - C) \\ &= i\left(C - C^\dagger\right)\end{aligned}$$

Arfken 3.4.9

The matrices A and B are both Hermitian: $A = A^\dagger$ and $B = B^\dagger$. The adjoint of their product is

$$\begin{aligned}(AB)^\dagger &= (AB)^\dagger \\ &= B^\dagger A^\dagger \\ &= BA\end{aligned}$$

For the product then to be Hermitian, we must have $AB = (AB)^{\dagger ger} = BA$, *i.e.* A and B must commute. Thus, this shows that *if* $AB = (AB)^\dagger$, *then* $[A, B] = 0$. To go the other way,

$$\begin{aligned}0 &= [A, B] \\ &= [A, B]^\dagger \\ &= (AB - BA)^\dagger \\ &= (AB)^\dagger - A^\dagger B^\dagger \\ &= (AB)^\dagger - AB\end{aligned}$$

and $AB = (AB)^\dagger$ and is Hermitian.

Arfken 3.4.12

(a) Two matrices U and H are related by

$$\begin{aligned} U &= e^{iaH} \\ &\equiv \mathbf{1} + iaH + \frac{(ia)^2}{2} H^2 + \frac{(ia)^3}{3!} H^3 + \cdots \end{aligned}$$

First assume $H = H^\dagger$ and take the adjoint of the above relation

$$U^\dagger = \mathbf{1}^\dagger - iaH^\dagger + \frac{(-ia)^2}{2!} \left(H^2\right)^\dagger + \frac{(-ia)^3}{3!} \left(H^3\right)^\dagger + \cdots$$

It should be clear that we need to show $(H^n)^\dagger = H^n$. Briefly, for $n = 1$ and $n = 2$, this is straightforward to show. By induction we can then demonstrate the general case. Taking it as a result, we have

$$\begin{aligned} U^\dagger &= \mathbf{1} - iaH + \frac{(ia)^2}{2!} H^2 - \frac{(ia)^3}{3!} H^3 + \cdots \\ &= e^{-iaH} \end{aligned}$$

By now multiplying on the left by $U = e^{iaH}$, we can see that $U^\dagger$ must equal U^{-1} and therefore U is unitary.

(b) Now assume $U^\dagger = U^{-1}$. We know that $U = e^{iaH}$. Its inverse is U^{-1} which we might guess is e^{-iaH}. But we need to show this:

$$\begin{aligned} U\,U^{-1} &= e^{iaH} e^{-iaH} \\ &= e^{iaH - iaH} \\ &= \mathbf{1} \end{aligned}$$

and we have verified it. Being unitary, we have

$$\begin{aligned} U^\dagger &= e^{-iaH} \\ &= \left(e^{iaH}\right)^\dagger \\ &= e^{-iaH^\dagger} \end{aligned}$$

where we have two (matrix-valued) Taylor series which are equal: $e^{-iaH} = e^{-iaH^\dagger}$. If the Taylor series are to be equal, we must have, in general, $H^n = (H^n)^\dagger = (H^\dagger)^n$. This will be true provided, $H = H^\dagger$, *i.e.* H is Hermitian.

Arfken 3.5.4

Assume the matrix A is *not* symmetric but that it can be diagonalized by an orthogonal similarity transformation. We then have

$$\begin{aligned} A'_{il} &= R_{ij} A_{jk} (R^T)_{kl} \\ &= R_{ij} A_{jk} R_{lk} \\ &= R_{lk} A_{jk} R_{ij} \end{aligned}$$

where R is the appropriate orthogonal matrix. Since A' is diagonal, it is symmetric: $A'_{il} = A'_{li}$. This implies

$$\begin{aligned} A'_{il} &= R_{ik} A_{jk} R_{lj} \\ &= R_{ik} A_{jk} (R^T)_{jl} \end{aligned}$$

Relabeling indices so that $k \to j$ and $j \to k$ above, it become clear that this can only equal the first line if $A_{jk} = A_{kj}$, *i.e.* A is symmetric. This is a contradiction and we conclude that a non-symmetric matrix cannot be diagonalized.

Arfken 3.5.8

Two matrices, A and B, are diagonalized by the same transformation:

$$A' = RAR^T$$
$$B' = RBR^T$$

These two diagonal matrices now commute:

$$\begin{aligned} 0 &= A'B' - B'A' \\ &= RAR^T RBR^T - RBR^T RAR^T \\ &= RABR^T - RBAR^T \\ &= R(AB - BA)R^T \end{aligned}$$

which will be true if and only if $AB = BA$.

Arfken 3.5.12

For a rigid body defined by $m_1 = 1$ at $(1, 1, -2)$, m_2 at $(-1, -1, 0)$, and m_3 at $(1, 1, 2)$, the components of the inertia matrix are

$$\begin{aligned} I_{xx} &= \sum_{i=1}^{3} m_i(r_i^2 - x_i^2) \\ &= m_1(r_1^2 - x_1^2) + m_2(r_2^2 - x_2^2) + m_3(r_3^2 - x_3^2) \\ &= 1 \cdot (6 - 1) + 2 \cdot (2 - 1) + 1 \cdot (6 - 1) \\ &= 12 \end{aligned}$$

with similar calculations leading to $I_{yy} = 12$, $I_{zz} = 8$, $I_{xy} = -4$, and $I_{xz} = I_{yz} = 0$. Putting it together,

$$I = \begin{pmatrix} 12 & -4 & 0 \\ -4 & 12 & 0 \\ 0 & 0 & 8 \end{pmatrix}$$

(b) Getting the eigenvalues and eigenvectors requires the secular equation

$$\begin{aligned} 0 &= \det(I - \lambda \mathbf{1}) \\ &= (8 - \lambda)\big((12 - \lambda)^2 - 16\big) \\ &= -(\lambda - 8)^2(\lambda - 16) \end{aligned}$$

Solving the eigenvalue equations for $\lambda = 16$ gives the equations $x = -y$ and $z = 0$ so we pick a normalized eigenvector of $(1, -1, 0)/\sqrt{2}$. The degenerate eigenvalue $\lambda = 8$ gives the equations $x = y$ and z can be anything. So one eigenvector associated with $\lambda = 8$ is $(1, 1, 1)/\sqrt{3}$. Another eigenvector which would go with the $\lambda = 8$ eigenvalue is $(-1, -1, 2)/\sqrt{6}$ which, one can readily check, is orthogonal to both the other eigenvectors.

Arfken 3.5.20

Diagonalize

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

The secular equation is

$$\begin{aligned} 0 &= \det(A - \lambda \mathbf{1}) \\ &= (1 - \lambda)\left((1 - \lambda)^2 - 1\right) \\ &= -\lambda(\lambda - 1)(\lambda - 2) \end{aligned}$$

with eigenvalues $\lambda = 0, 1, 2$. The eigenvector associated with the first eigenvalue can be found from the equations $x = 0$ and $y + z = 0$. It is $(0, 1, -1)/\sqrt{2}$. For the second eigenvalue, the eigenvector can be determined from the equations $z = 0$ and $y = 0$ with x anything. The second eigenvector is thus $(1, 0, 0)$. For $\lambda = 2$, the equations for the eigenvector are $x = 0$ and $y = z$. Thus we have $(0, 1, 1)/\sqrt{2}$.

Arfken 3.5.27

Diagonalize

$$A = \begin{pmatrix} 5 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \end{pmatrix}$$

The secular equation is

$$\begin{aligned} 0 &= \det(A - \lambda \mathbf{1}) \\ &= (5-\lambda)(1-\lambda)(2-\lambda) + 2(1-\lambda)\cdot 2 \\ &= (1-\lambda)\left[\lambda^2 - 7\lambda + 6\right] \\ &= -(\lambda-1)^2(\lambda-6) \end{aligned}$$

with eigenvalues $\lambda = 1, 1, 6$. The eigenvector associated with the last eigenvalue $\lambda = 6$ can be found from the equations $y = 0$ and $x = 2z$. It is $(2, 0, 1)/\sqrt{5}$. For the degenerate eigenvalue, the eigenvectors can be determined from the equations $2x = -z$ and with y anything. Thus one eigenvector is $(1, 0, -2)/\sqrt{6}$. To get another, just notice that $(0, 1, 0)$ satisfies the equations and is orthogonal to the other two eigenvectors.

Arfken 3.6.3

The secular equation for the matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$\begin{aligned} 0 &= \det(A - \lambda \mathbf{1}) \\ &= (a-\lambda)(d-\lambda) - bc \\ &= \lambda^2 - \lambda(a+d) + ad - bc \\ &= \lambda^2 - \lambda \mathrm{Tr} A + \det A \end{aligned}$$

Arfken 3.6.7

In bra-ket notation

$$A|r_i\rangle = \lambda_i|r_i\rangle \tag{1}$$
$$A|r_j\rangle = \lambda_j|r_j\rangle \tag{2}$$

Taking the adjoint of Eq. (2), we get

$$(A|r_j\rangle)^\dagger = (\lambda_j|r_j\rangle)^\dagger$$
$$\langle r_j|A^\dagger = \lambda^*\langle r_j| \tag{3}$$

Now multiply by $\langle r_j|$ on the left of Eq. (1) and by $|r_i\rangle$ on the right of Eq.(3). Finally, subtract the two

$$\langle r_j|A - A^\dagger|r_i\rangle = (\lambda_i - \lambda_j^*)\langle r_j|r_i\rangle$$

The left hand side is zero for a Hermitian matrix ($A = A^\dagger$). For $i \neq j$ (and no degeneracy) the eigenvectors are orthogonal. For $i = j$, the eigenvalues must be real: $\lambda_i = \lambda_i^*$.

Now take Eq. (1) and multiply by A^{-1}

$$A^{-1}A|r_i\rangle = \lambda_i A^{-1}|r_i\rangle$$

which can be re-written

$$A^{-1}|r_i\rangle = \frac{1}{\lambda_i}|r_i\rangle$$

Multiply this by $\langle r_j|$ on the left and subtract from this Eq. (3)

$$\langle r_j|A^{-1} - A^\dagger|r_i\rangle = (\frac{1}{\lambda_i} - \lambda_j^*)\langle r_j|r_i\rangle$$

For a unitary matrix, the left hand side is zero and for $i = j$, we must have

$$\lambda_i\lambda_i^* = 1$$

Thus if a matrix is both Hermitian and unitary, $\lambda_i\lambda_i = 1$ and the eigenvalues can only be ± 1.

Arfken 3.6.14

We have

$$A = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 2 \\ 1 & -4 \end{pmatrix}$$

The transpose, $\tilde{A}$ together with $A\tilde{A}$ and $\tilde{A}A$ are

$$\tilde{A} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix}$$
$$A\tilde{A} = \frac{1}{5}\begin{pmatrix} 8 & -6 \\ -6 & 17 \end{pmatrix}$$
$$\tilde{A}A = \frac{1}{5}\begin{pmatrix} 5 & 0 \\ 0 & 20 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

(b) The eigenvalues of $A\tilde{A}$ come out of the secular equation:

$$\begin{aligned}
0 &= \det(A\tilde{A} - \lambda\mathbf{1}) \\
&= (\frac{8}{5} - \lambda)(\frac{17}{5} - \lambda) - \frac{36}{25} \\
&= (\lambda - 4)(\lambda - 1)
\end{aligned}$$

Thus, $\lambda_n^2 = 1, 4$. The eigenvectors, $|g_n\rangle$, associated with these are $(2,1)/\sqrt{5}$ and $(1,-2)/\sqrt{5}$, respectively.

(c) The eigenvalues of $\tilde{A}A$ are simple since it is a diagonal matrix. They are as before: $\lambda_n^2 = 1, 4$. However, the eigenvectors, $|f_n\rangle$, are $(1,0)$ and $(0,1)$.

(d) Note that

$$\begin{aligned}
A|f_1\rangle &= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 2 \\ 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \lambda_1|g_1\rangle \\
A|f_2\rangle &= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 2 \\ 1 & -4 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -4 \end{pmatrix} = \lambda_2|g_2\rangle \\
\tilde{A}|g_1\rangle &= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 5 \\ 0 \end{pmatrix} = \lambda_1|f_1\rangle \\
\tilde{A}|g_2\rangle &= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 1 \\ 2 & -4 \end{pmatrix} \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 0 \\ 10 \end{pmatrix} = \lambda_2|f_2\rangle
\end{aligned}$$

(e) By construction, we find

$$\begin{aligned}
A &= \sum_n \lambda_n |g_n\rangle\langle f_n| \\
&= \lambda_1|g_1\rangle\langle f_1| + \lambda_2|g_2\rangle\langle f_2| \\
&= 1 \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \end{pmatrix} + 2 \cdot \frac{1}{\sqrt{5}}\begin{pmatrix} 2 \\ -4 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \\
&= \frac{1}{\sqrt{5}}\begin{pmatrix} 2 & 2 \\ 1 & -4 \end{pmatrix}
\end{aligned}$$