Assignment 12

Arfken 4.1.1 Show that the group of $n \times n$ orthogonal matrices ($O(n)$) has $n(n-1)/2$ independent parameters.

First we need to realize that a general orthogonal matrix, O, has n^2 elements or parameters, not all of which are independent. The condition that it be orthogonal is $O\,O^T = \mathbf{1}$, or in component notation,

$$\begin{aligned}\delta_{ik} &= O_{ij}\left(O^T\right)_{jk} \\ &= O_{ij}O_{kj} \\ &= O_{kj}O_{ij}\end{aligned}$$

thus the product of an orthogonal matrix and its transpose must by symmetric. This says that all the equations implied by the above matrix condition are not independent. There are n equations which come from the diagonal terms that must equal 1. However, the off diagonal terms in the upper right corner of the matrix are identical to those off-diagonal terms in the lower left corner of the matrix, so the resulting conditions are identical. The number of these conditions is $(n^2 - n)/2$ (half the total number of matrix elements minus the diagonal terms).* Thus the total number of conditions from the orthogonality conditions are $n + (n^2 - n)/2 = (n^2 + n)/2$ and so the total number of *independent* parameters of a general orthogonal matrix is $n^2 - (n^2 + n)/2 = n(n-1)/2$.

Arfken 4.1.2 Show that the group of $n \times n$ unitary matrices ($U(n)$) has $n^2 - 1$ independent parameters.

As with the previous problem, we must realize that a general $n \times n$ unitary matrix, U, has $2n^2$ parameters since each entry is complex and counts twice. The condition that the matrix be unitary is $U\,U^\dagger = \mathbf{1}$ or

$$\begin{aligned}\delta_{ik} &= U_{ij}\left(U^\dagger\right)_{jk} \\ &= U_{ij}U^*_{kj} \\ &= U_{kj}U^*_{ij}\end{aligned}$$

showing that the product of a unitary matrix and its adjoint must be Hermitian. Again, this says that the resulting equations implied in the above matrix condition are not all independent. Being Hermitian, each of the diagonal components of $A \equiv UU^\dagger$ must be real and equal to one. This gives n conditions. The Hermiticity of $A \equiv UU^\dagger$ means that off-diagonal components in the upper right of the matrix are complex conjugates of the off-diagonal components in the lower left of the matrix. Thus these conditions are not independent, but in fact, the same. The number of these conditions is twice that of the real orthogonal case: $2 \cdot (n^2 - n)/2$.

The total number of conditions is now $n + (n^2 - n) = n^2$ and the total number independent parameters is $2n^2 - n^2 = n^2$. Of course this seems wrong. However, there is an error in the statement of the problem. The problem ostensibly asks for the parameters describing the group $U(n)$, but I am almost certain they *intended* to ask about $SU(n)$, the group of all unitary matrices with determinant equal to one (sometimes called unimodular). In this case there is an extra (unstated) condition, that of $\det U = 1$. With this condition, the number of independent parameters is now $n^2 - 1$.

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* We could also have gotten this number by adding up the number of independent terms in each row of the matrix:

$$1 + 2 + 3 + \cdots + n - 1 = \frac{1}{2}\,(n-1)n.$$

Arfken 4.1.3 Show that the matrix representation of $SL(2)$ forms a group.

A general element, A, of $SL(2)$ will be

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c and d are complex and such that $ad - bc = 1$. To show that $SL(2)$ is a group, we must assure ourselves that this and like elements satisfy the group axioms with matrix multiplication being our putative group operation. Closure implies that if A_1 and A_2 are elements of the group, their product will be as well:

$$\begin{aligned} A_1 \cdot A_2 &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \\ &= \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1b_2 & c_1b_2 + d_1d_2 \end{pmatrix} \end{aligned}$$

which is a complex 2×2 matrix since each of its elements is complex. However, we must check that its determinant is 1. This is most easily done by taking the determinant

$$\begin{aligned} \det\left(A_1 \cdot A_2\right) &= \det A_1 \cdot \det A_2 \\ &= 1 \end{aligned}$$

where we have used the fact that since A_1 and A_2 are elements of the group, their individual determinants must be 1.

Associativity is shown after some algebra:

$$\begin{aligned} \left(A_1 \cdot A_2\right) A_3 &= \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1b_2 & c_1b_2 + d_1d_2 \end{pmatrix} \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix} \\ &= \begin{pmatrix} (a_1a_2 + b_1c_2)a_3 + (a_1b_2 + b_1d_2)c_3 & (a_1a_2 + b_1c_2)b_3 + (a_1b_2 + b_1d_2)d_3 \\ (c_1a_2 + d_1b_2)a_3 + (c_1b_2 + d_1d_2)c_3 & (c_1a_2 + d_1b_2)b_3 + (c_1b_2 + d_1d_2)d_3 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} A_1\left(A_2 \cdot A_3\right) &= \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2a_3 + b_2c_3 & a_2b_3 + b_2d_3 \\ c_2a_3 + d_2b_3 & c_2b_3 + d_2d_3 \end{pmatrix} \\ &= \begin{pmatrix} a_1(a_2a_3 + b_2c_3) + b_1(c_2a_3 + d_2b_3) & a_1(a_2b_3 + b_2d_3) + b_1(c_2b_3 + d_2d_3) \\ c_1(a_2a_3 + b_2c_3) + d_1(c_2a_3 + d_2b_3) & c_1(a_2b_3 + b_2d_3) + d_1(c_2b_3 + d_2d_3) \end{pmatrix} \end{aligned}$$

which is equal to the first.

The inverse is found by recalling from Chapter 3 the inverse for a 2×2 matrix

$$A^{-1} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Note that $A \cdot A^{-1} = \mathbf{1}$ as required and that the elements of A^{-1} are indeed complex. Finally, $\det\left(A^{-1}\right) = (\det A)^{-1} = 1$. However, we have not yet proved a *unique* inverse. This is done by assuming there is another inverse, B. If that is the case, then in addition to $A \cdot A^{-1} = \mathbf{1}$, we have $A \cdot B = \mathbf{1}$ such that B is an inverse. Now multiply that last equation by A^{-1} on the left. We get $A^{-1}A \cdot B = A^{-1}$, *i.e.* $B = A^{-1}$ and A^{-1} is unique.

The identity element of the group is "obviously" the unit matrix

$$\mathbf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as can be seen by multiplying this by A and getting A back. But again, we have the question of the *unique* identity. In a similar argument as before, assume B is another identity element, then $A \cdot B = A$. Multiply on the left by A^{-1}, the inverse of A, and get $A^{-1}A \cdot B = A^{-1}A = \mathbf{1}$ and we get that $B = \mathbf{1}$ and $\mathbf{1}$ is the unique identity element.

Arfken 4.2.2 Find the general form of a 2×2 unitary, unimodular matrix.

A general 2×2 complex matrix can be written as

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where a, b, c, and d are complex. That it must be unitary means that we must have $U^{-1} = U^\dagger$, or

$$\frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}$$

As unimodular simply means the determinant must be one, we can let $ad - bc = 1$ and we have the relations

$$d = a^*$$
$$c = -b^*$$

so that

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix}$$

In this case, we also must have $|a|^2 + |b|^2 = 1$.

Arfken 3.2.7 We have three matrices

$$A = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$$

Their products are

$$\begin{aligned} AA &= \mathbf{1}, & AB &= C, & AC &= B \\ BA &= C, & BB &= \mathbf{1}, & BC &= A \\ CA &= B, & CB &= A, & CC &= \mathbf{1} \end{aligned}$$

Arfken 4.7.2 Rotations by π, reflections and inversions are represented by A_π, m and i respectively. Their combinations (multiplications), when including the identity, $\mathbf{1}$, are governed by the multiplication table

	$\mathbf{1}$	A_π	m	i
$\mathbf{1}$	$\mathbf{1}$	A_π	m	i
A_π	A_π	$\mathbf{1}$	i	m
m	m	i	$\mathbf{1}$	A_π
i	i	m	A_π	$\mathbf{1}$

On comparison with the multiplication table for the vierergruppe (in Arfken Table 4.3), we see that they are identical with the replacements $A_\pi \to V_1$, $m \to V_2$ and $i \to V_3$. In consequence, the two groups are the same, i.e. they are isomorphic.

Arfken 4.7.5 Explore the vierergruppe.
(a) We have the multiplication table (again) for the vierergruppe

	$\mathbf{1}$	A	B	C
$\mathbf{1}$	$\mathbf{1}$	A	B	C
A	A	$\mathbf{1}$	C	B
B	B	C	$\mathbf{1}$	A
C	C	B	A	$\mathbf{1}$

To find the classes, we need to know the mutually conjugate elements of the group. Note that every element is its own inverse and that the group is Abelian or commutative. For the element A, we have

$$AAA^{-1} = A$$
$$BAB^{-1} = CB = A$$
$$CAC^{-1} = BC = A$$

Without doing the others, we note that for this group, if g and X are elements of this group, then $gXg^{-1} = gg^{-1}X = X$ and each element is only self-conjugate. Thus there are as many classes as there are group elements, in this case four. Each class has only one element of the group in it.

(b) The character of a class is given by $\chi(\{X\}) = \mathrm{tr}(X)$. We have

$$\chi(\{\mathbf{1}\}) = 2$$
$$\chi(\{A\}) = -2$$
$$\chi(\{B\}) = 0$$
$$\chi(\{C\}) = 0$$

Note that two different classes may have the same character (in this case, 0).

(c) With regard to subgroups, because each member of the group its own inverse, each member of the group that is not the identity when paired with the identity must form a subgroup. For example, $\{\mathbf{1}, A\}$ forms a subgroup as closure is satisfied, associativity is satisfied, there is an identity element and an inverse. Hence there are three two-element subgroups of the vierergruppe, namely

$$\{\mathbf{1}, A\}, \quad \{\mathbf{1}, B\}, \quad \{\mathbf{1}, C\}$$

Arfken 4.7.6 Explore the 2×2 matrix representation of the cyclic group, C_4.

The matrices are given by

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

together with the identity, the unit matrix. The multiplication table is

	$\mathbf{1}$	A	B	C
$\mathbf{1}$	$\mathbf{1}$	A	B	C
A	A	B	C	$\mathbf{1}$
B	B	C	$\mathbf{1}$	A
C	C	$\mathbf{1}$	A	B

This group, as can be seen, is also Abelian. However, only B (and the identity) is its own inverse. Indeed, A and C are inverses of each other.

(a) Finding mutually conjugate elements of the group, we have

$$AAA^{-1} = A$$
$$BAB^{-1} = CB = A$$
$$CAC^{-1} = \mathbf{1}A = A$$

and A is in a class by itself. Doing this for B, we get

$$ABA^{-1} = CC = B$$
$$BBB^{-1} = B$$
$$CBC^{-1} = AA = B$$

and B is in a class by itself. For C, we find

$$ACA^{-1} = \mathbf{1}C = C$$
$$BCB^{-1} = AB = C$$
$$CCC^{-1} = BA = C$$

and we round out the set. Thus there are four classes each with one of the group elements.

(b) The character, $\chi(X)$, is

$$\chi(\mathbf{1}) = 2$$
$$\chi(A) = 0$$
$$\chi(B) = -2$$
$$\chi(C) = 0$$

(c) In terms of subgroups, in looking at the multiplication table, there are no blocks that would suggest a subgroup. However, we do know that B is its own inverse. Hence there is one subgroup, namely $\{\mathbf{1}, B\}$. And that's it. We might be tempted to try $\{\mathbf{1}, A, C\}$ but because $A^2 = B$, for example, that won't work.