## Assignment 13

**Arfken 4.2.1** First show that the Pauli matrices are the generators of $SU(2)$ without using the general parametrization of the general unitary $2 \times 2$ matrix.

Instead of working with the unitary matrices, let's work with the generators themselves in order to see if we can deduce them from general properties. We know that the generators must by Hermitian, traceless, $2 \times 2$ and that there should be $n^2 - 1 = 3$ of them (since the number of generators of a group is equivalent to the number of free parameters describing a general matrix in the group). Such a matrix might be

$$H = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

The Hermiticity condition for the generators, namely that $H = H^\dagger$, imply that $a$ and $d$ are real and that $c$ and $b$ are complex conjugates of each other. Tracelessness says that $a = -d$. Such a matrix can now be parametrized by

$$H = \begin{pmatrix} a & b_1 - ib_2 \\ b_1 + ib_2 & -a \end{pmatrix}$$

with three free parameters. Setting each value in turn to one while the others are zero gives us a basis of generators

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

which are indeed the Pauli spin matrices, $\sigma_z$, $\sigma_x$, and $\sigma_y$.

Doing a similar thing for $SU(3)$, we have

$$H = \begin{pmatrix} a & b & c \\ l & m & n \\ x & y & z \end{pmatrix}$$

where each entry is complex. The conditions for the generators now say that the diagonal elements, $a$, $m$, and $z$ are all real and sum to zero. The off-diagonal elements are complex conjugates of their transposed elements. Our general generator is now

$$H = \begin{pmatrix} a & b_1 - ib_2 & c_1 - ic_2 \\ b_1 + ib_2 & -a - z & n_1 - in_2 \\ c_1 + ic_2 & n_1 + in_2 & z \end{pmatrix}$$

which has eight free (real) parameters as demanded by $n^2 - 1$. Again, setting each value in turn to one while the others are zero gives us a basis of generators

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \lambda_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which, except for the last one, are exactly the generators given in the text on page 258 for $SU(3)$. The first seven are called $\lambda_i$ with $i$ ranging from 1 to 7. The generator $\lambda_8$ in the text is obtained by using the normalization given in the problem: $\mathrm{Tr}(\lambda_i \lambda_j) = 2\delta_{ij}$. Alternatively, we can get Arfken's version of $\lambda_8$ from our matrices by subtracting twice our last matrix from the first and dividing by $\sqrt{3}$.

At this point, you are to find the structure constants for $SU(3)$. This is a tedious part left to the diligent student.

**(c)** The Casimir invariant for $SU(3)$, in analogy with $\sigma_1^2+\sigma_2^2+\sigma_3^2$ for $SU(2)$ is

$$\lambda_1^2+\lambda_2^2+\lambda_3^2+\lambda_4^2+\lambda_5^2+\lambda_6^2+\lambda_7^2+\lambda_8^2 = \frac{16}{3}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

where we have used

$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

**Arfken 4.2.3** Determine three $SU(2)$ subgroups of $SU(3)$.

We need only work with the generators. Using the definitions of the Pauli matrices and the $\lambda_i$ generators, $\lambda_i$, from 4.2.1, we can make the following observation (already made in the text):

$$\lambda_1 = \begin{pmatrix}\sigma_x & 0\\ 0 & 0\end{pmatrix}, \quad \lambda_2 = \begin{pmatrix}\sigma_y & 0\\ 0 & 0\end{pmatrix}, \quad \lambda_3 = \begin{pmatrix}\sigma_z & 0\\ 0 & 0\end{pmatrix},$$

where, hopefully, the extra 0s in the above $3\times 3$ matrices make sense. Importantly, note that every product of these three matrices amongst themselves produces one of them (perhaps multiplied by a constant scalar factor) or the matrix

$$\begin{pmatrix}\mathbf{1} & 0\\ 0 & 0\end{pmatrix}.$$

Said another way, there is no combination of these four matrices that leads to any of the other $\lambda_i$ matrices. While this set of four matrices are not elements of a group, they are some of the generators of $SU(3)$. In particular, they form a subset of those generators and when only this set of generators are exponentiated they create elements of the group $SU(3)$ which are themselves a subgroup of $SU(3)$.

To see this, note that the elements of this subgroup take the general form

$$U = \exp\Big[i\sum_{i=1}^{3}\epsilon_i\lambda_i\Big]$$

where the $\epsilon_i$s are arbitrary constants. The actual Taylor series defining the exponentiation will be a complicated mess involving powers of the combination of generators given by $\sum\epsilon_i\lambda_i$, but the point is that none of those powers will ever give a $3\times 3$ matrix which has anything but a nonzero block of elements in the upper left hand corner. The third row and column will always contain only 0s. Nevertheless, the resulting group elements retain their unitary and unimodular character. Including the identity with them, it is clear that we have a set of matrices that remain closed under multiplication. We have a subgroup. It should be obvious that the subgroup is $SU(2)$ with a funny representation by $3\times 3$ matrices. Nonetheless, it is $SU(2)$.

We can repeat this process two more times with the following sets of generators:

$$\{\lambda_6,\lambda_7,\lambda_8\} \quad \text{and} \quad \{\lambda_4,\lambda_5,\lambda_3-\lambda_8\}$$

where we have used our definition (and not Arfken's) above n 4.2.1 for $\lambda_8$. Both of these sets produce combinations among themselves that never leave the set. The resulting exponentiated combinations produce group elements that likewise form subgroups that are $SU(2)$.

**Arfken 4.2.4** Find the translation operator $T(a)$ that converts $\psi(x)$ into $\psi(x+a)$.

Starting with a Taylor series, we can write

$$\begin{aligned}\psi(x+a) &= \psi(x) + a\,\psi'(x) + \frac{a^2}{2}\,\psi''(x) + \cdots\\ &= \left\{1 + a\,\frac{d}{dx} + \frac{a^2}{2}\,\frac{d^2}{dx^2} + \cdots\right\}\psi(x)\\ &= \left\{1 + iap_x + i^2\frac{a^2}{2}\,p_x^2 + \cdots\right\}\psi(x)\\ &= e^{iap_x}\,\psi(x)\end{aligned}$$

where we have used the momentum operator $p_x = -id/dx$. We can now identify $T(a) = e^{iap_x}$.

**Arfken 4.2.5** Consider the general $SU(2)$ element to be built up of three Euler rotations.

From Eq. 4.38 we see that a generic element of $SU(2)$ can be written as

$$U(\xi, \eta, \zeta) = \begin{pmatrix} e^{i\xi}\cos\eta & e^{i\zeta}\sin\eta \\ -e^{-i\zeta}\sin\eta & e^{-i\xi}\cos\eta \end{pmatrix}$$

This is what we want to arrive at. The point of the problem is to build this up from three (Euler) rotations. The first is a rotation of $a/2$ about the $z$-axis. This is produced by the unitary operator

$$U_1 = e^{ia\sigma_z/2}$$

while the subsequent rotations are $b/2$ about the $x$-axis followed by another about the $z$-axis of $c/2$. The combined operator is thus

$$U = U_3U_2U_1 = e^{ic\sigma_z/2}e^{ib\sigma_x/2}e^{ia\sigma_z/2}$$

where, of course, the $\sigma_i$ are the Pauli matrices.

It is now worthwhile to see what these rotations are. In particular,

$$\begin{aligned} U_1 = e^{ia\sigma_z/2} &= \mathbf{1} + \frac{ia}{2}\sigma_z + \frac{1}{2!}\left(\frac{ia}{2}\right)^2\sigma_z^2 + \frac{1}{3!}\left(\frac{ia}{2}\right)^3\sigma_z^3 + \cdots \\ &= \mathbf{1}\sum_{n=0}^{\infty}\frac{1}{(2n)!}\left(\frac{ia}{2}\right)^{2n} + \sigma_z\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left(\frac{ia}{2}\right)^{2n+1} \\ &= \mathbf{1}\,\cos\frac{a}{2} + i\sigma_z\,\sin\frac{a}{2} \\ &= \begin{pmatrix} e^{ia/2} & 0 \\ 0 & e^{-ia/2} \end{pmatrix} \end{aligned}$$

A similar calculation will hold for the $c/2$ rotation. Likewise, one can show that

$$\begin{aligned} U_2 = e^{ib\sigma_x/2} &= \mathbf{1}\,\cos\frac{b}{2} + i\sigma_x\,\sin\frac{b}{2} \\ &= \begin{pmatrix} \cos(b/2) & i\sin(b/2) \\ i\sin(b/2) & \cos(b/2) \end{pmatrix} \end{aligned}$$

Combining these now, we get

$$\begin{aligned} U &= U_3U_2U_1 \\ &= \begin{pmatrix} e^{ic/2} & 0 \\ 0 & e^{-ic/2} \end{pmatrix}\begin{pmatrix} \cos(b/2) & i\sin(b/2) \\ i\sin(b/2) & \cos(b/2) \end{pmatrix}\begin{pmatrix} e^{ia/2} & 0 \\ 0 & e^{-ia/2} \end{pmatrix} \\ &= \begin{pmatrix} e^{i(a+c)/2}\cos(b/2) & ie^{-i(a-c)/2}\sin(b/2) \\ ie^{i(a-c)/2}\sin(b/2) & e^{-i(a+c)/2}\cos(b/2) \end{pmatrix} \\ &= \begin{pmatrix} e^{i(a+c)/2}\cos(b/2) & e^{-i(a-c)/2+i\pi/2}\sin(b/2) \\ -e^{i(a-c)/2-i\pi/2}\sin(b/2) & e^{-i(a+c)/2}\cos(b/2) \end{pmatrix} \end{aligned}$$

Making the straight across comparison, we note that

$$\begin{aligned} \xi &= \frac{1}{2}(a+c) \\ \zeta &= \frac{1}{2}(c-a) + \frac{\pi}{2} \\ \eta &= \frac{b}{2} \end{aligned}$$

Inverting, we get

$$\begin{aligned} a &= \xi + \zeta - \frac{\pi}{2} \\ b &= 2\eta \\ c &= \xi - \zeta + \frac{\pi}{2} \end{aligned}$$

which is what was asked for.