Assignment 1

Arfken 1.1.6 Show $\vec{A} = \vec{B}$ is three scalar equations.

Write $\vec{A} = \vec{B}$ in a coordinate system as

$$\hat{\imath}A_x + \hat{\jmath}A_y + \hat{k}A_z = \hat{\imath}B_x + \hat{\jmath}B_y + \hat{k}B_z$$

Since the basis vectors $\hat{\imath}, \hat{\jmath}, \hat{k}$ are linearly independent, the components of each on one side must equal the corresponding components on the other. Hence:

$$\begin{aligned} A_x &= B_x \\ A_y &= B_y \\ A_z &= B_z \end{aligned}$$

and we have three *scalar* equations.

Arfken 1.2.1 (a) Show that the magnitude of a vector $\vec{A}$ is invariant.

In one coordinate system, $\vec{A}$ is (A_x, A_y). In a rotated (transformed) system, it is (A'_x, A'_y). The magnitude of $\vec{A}$ in the first is $|A| = (A_x{}^2 + A_y{}^2)^{1/2}$. Assume in the rotated system, the magnitude in the transformed coordinates is

$$\begin{aligned} |A'| &= (A'_x{}^2 + A'_y{}^2)^{1/2} \\ &= \left[(A_x \cos\varphi + A_y \sin\varphi)^2 + (-A_x \sin\varphi + A_y \cos\varphi)^2\right]^{1/2} \\ &= \big[A_x{}^2 \cos^2\varphi + 2A_xA_y \cos\varphi \sin\varphi + A_y{}^2 \sin^2\varphi \\ &\qquad + A_x{}^2 \sin^2\varphi - 2A_xA_y \cos\varphi \sin\varphi + A_y{}^2 \cos^2\varphi\big]^{1/2} \\ &= \left[A_x{}^2 + A_y{}^2\right]^{1/2} \\ &= |A| \end{aligned}$$

and the magnitude is invariant (*i.e.* it's a scalar) under coordinate transformations (rotations).

(b) Show that $\vec{A} = \vec{A}'$ defines the same direction in space.

To begin, it is important to understand what this innocuous seeming statement is saying. It is that the vector is the same vector regardless of the coordinate system in which it is expressed. Note that we can write both vector expressions as

$$\begin{aligned} \vec{A} &= A_x\hat{\imath} + A_y\hat{\jmath} \\ &= |A| \cos\alpha\, \hat{\imath} + |A| \sin\alpha\, \hat{\jmath} \\ \vec{A}' &= A'_x\hat{\imath}' + A'_y\hat{\jmath}' \\ &= |A'| \cos\alpha'\, \hat{\imath}' + |A'| \sin\alpha'\, \hat{\jmath}' \end{aligned}$$

We have the same vector in two different coordinate systems. Note that both the components and the basis vectors are different. However, if we use the relation between the basis vectors

$$\begin{aligned} \hat{\imath}' &= \hat{\imath} \cos\varphi + \hat{\jmath} \sin\varphi \\ \hat{\jmath}' &= -\hat{\imath} \sin\varphi + \hat{\jmath} \cos\varphi \end{aligned}$$

and the fact that the magnitude of the vector must be the same in both coordinate systems, we can write

$$\begin{aligned} |A|\big(\hat{\imath} \cos\alpha + \hat{\jmath} \sin\alpha\big) &= |A'|\big(\hat{\imath}' \cos\alpha' + \hat{\jmath}' \sin\alpha'\big) \\ &= |A'|\Big[\hat{\imath}\big(\cos\alpha' \cos\varphi - \sin\alpha \sin\varphi\big) + \hat{\jmath}\big(\sin\alpha' \cos\varphi + \cos\alpha' \sin\varphi\big)\Big] \\ &= |A'|\big(\hat{\imath} \cos(\alpha' + \varphi) + \hat{\jmath} \sin(\alpha' + \varphi)\big) \end{aligned}$$

Because the unit vectors are independent, we have that

$$\cos\alpha = \cos(\alpha' + \varphi)$$
$$\sin\alpha = \sin(\alpha' + \varphi)$$

or $\alpha' = \alpha - \varphi$. This implies what we wanted, namely that the expression $\vec{A} = \vec{A}'$ defines the same direction in space.

Arfken 1.3.3 Find the surface swept out by $\vec{r}$ if (a) $(\vec{r} - \vec{a}) \cdot \vec{a} = 0$ and (b) $(\vec{r} - \vec{a}) \cdot \vec{r} = 0$.

Let $\vec{r} = (x, y, z)$ and $\vec{a} = (a_1, a_2, a_3)$. Then we have for (a):

$$(\vec{r} - \vec{a}) \cdot \vec{a} = a_1(x - a_1) + a_2(y - a_2) + a_3(z - a_3)$$

which is the plane perpendicular to the vector $\vec{a}$ and passing through the point (a_1, a_2, a_3).

Likewise, for (b):

$$\begin{aligned}(\vec{r} - \vec{a}) \cdot \vec{r} &= (x - a_1)x + (y - a_2)y + (z - a_3)z \\ &= \left(x - \frac{a_1}{2}\right)^2 + \left(y - \frac{a_2}{2}\right)^2 + \left(z - \frac{a_3}{2}\right)^2 - \frac{1}{4}\left({a_1}^2 + {a_2}^2 + {a_3}^2\right)\end{aligned}$$

which, when set to zero, is the usual equation for a sphere centered at $\frac{1}{2}(a_1, a_2, a_3)$ with radius one half the magnitude of $\vec{a}$.

Arfken 1.4.8 Find some trigonometric identities.

First note that each of the three vectors, $\vec{P}, \vec{Q}, \vec{R}$ has a magnitude of one. Also note that $\vec{P}$ is the unit vector drawn from the origin to the point $(x, y, 0)$ and makes an angle of θ with the x-axis. Similarly, $\vec{Q}$ is the unit vector extending from the origin to a point below the x-axis ($y < 0$). Thinking of it another way, it is a vector rotated an angle φ below the x-axis. We only need these two now. Calculate

$$\begin{aligned}\vec{P} \cdot \vec{Q} &= |\vec{P}| \cdot |\vec{Q}| \cos(\theta + \varphi) \\ &= \cos(\theta + \varphi) \\ &= \cos\theta\cos\varphi - \sin\theta\sin\varphi\end{aligned}$$

where in the last line we have used the component definition of the dot product.

Likewise, we have

$$\begin{aligned}|\vec{P} \times \vec{Q}| &= |\vec{P}| \cdot |\vec{Q}| \sin(\theta + \varphi) \\ &= \sin(\theta + \varphi) \\ &= \sin\theta\cos\varphi + \sin\varphi\cos\theta\end{aligned}$$

Arfken 1.5.5 Find an expression for the angular momentum in terms of angular velocity.

$$\begin{aligned}\vec{L} &= \vec{r} \times \vec{p} \\ &= m\,\vec{r} \times \vec{v} \\ &= m\,\vec{r} \times (\vec{\omega} \times \vec{r}) \\ &= m\,\left[\vec{\omega}\,(\vec{r} \cdot \vec{r}) - \vec{r}\,(\vec{r} \cdot \vec{\omega})\right] \\ &= m\,r^2\left[\vec{\omega} - \hat{r}\,(\hat{r} \cdot \omega)\right]\end{aligned}$$

where we have used the "$BAC - CAB$" rule. Or, if you want to use index notation,

$$\begin{aligned}L_i &= m\,\epsilon_{ijk} r_j (\epsilon_{klm} \omega_l r_m) \\ &= m\,\epsilon_{kij}\,\epsilon_{klm} \cdot r_j \omega_l r_m \\ &= m\,\left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) r_j \omega_l r_m \\ &= m\,\left(\omega_i r_j r_j - r_i r_j \omega_j\right)\end{aligned}$$

which gives exactly as above when put back into the "usual" vector notation.

Arfken 1.5.18 Consider the force(s) due to moving charges.

The magnetic induction due to a single moving charge is

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q_1}{r^2}\vec{v}_1 \times \hat{r}$$

Recall that the force on another charge, q_2, due to a magnetic induction is

$$\vec{F}_2 = q_2\vec{v}_2 \times \vec{B}_1$$

where the 1 has been appended to $\vec{B}_1$ to emphasize that it is the induction due to the first charge, q_1. Putting this together, we get

$$\vec{F}_2 = \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^2}\vec{v}_2 \times (\vec{v}_1 \times \hat{r})$$

Correspondingly, the force on charge, q_1, due to the field produced by q_2 is

$$\begin{aligned}\vec{F}_1 &= q_1\vec{v}_1 \times \vec{B}_2 \\ &= q_1\vec{v}_1 \times \left(\frac{\mu_0}{4\pi}\frac{q_2}{r^2}\vec{v}_2 \times (-\hat{r})\right) \\ &= -\frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^2}\vec{v}_1 \times (\vec{v}_2 \times \hat{r})\end{aligned}$$

where we have used $-\hat{r}$ to emphasize that the force now points from q_2 *to* q_1.

In the event the two charges are moving in parallel directions and with the same speed: $|\vec{v}_1| = |\vec{v}_2| = v$, $\vec{F}_1$ becomes

$$\begin{aligned}\vec{F}_1 &= -\frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^2}\left(\vec{v}_2(\vec{v}_2 \cdot \hat{r}) - \hat{r}(\vec{v}_1 \cdot \vec{v}_2)\right) \\ &= \frac{\mu_0}{4\pi}\frac{q_1 q_2}{r^2}v^2\hat{r}\end{aligned}$$

Similarly, $\vec{F}_2$ gives the same thing with an extra negative sign.

Arfken 1.6.2 Find the normal and tangent plane to a sphere.

To form the normal vector, take the gradient of the function $f(x, y, z) = x^2 + y^2 + z^2 - 3$ and then divide by the magnitude of the gradient. We get

$$\begin{aligned}\frac{\vec{\nabla} f}{|\vec{\nabla} f|} &= \frac{\hat{\imath}x + \hat{\jmath}y + \hat{k}z}{\sqrt{x^2 + y^2 + z^2}} \\ &= \frac{\hat{\imath} + \hat{\jmath} + \hat{k}}{\sqrt{3}}\end{aligned}$$

where in the last line we have evaluated at the point $(1, 1, 1)$.

Now consider the plane tangent to the sphere at the same point. It's equation will be

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

where the point (x_0, y_0, z_0) is the point of tangency, $(1, 1, 1)$. The coefficients can be determined by calculating the normal to the plane at the point of tangency and equating this to the normal to the sphere at the same point.

$$\frac{\vec{\nabla} f}{|\vec{\nabla} f|} = \frac{\hat{\imath}a + \hat{\jmath}b + \hat{k}c}{\sqrt{a^2 + b^2 + c^2}}$$

Equating coefficients of the corresponding unit vectors, it is now straightforward to show that $a = b = c = 1$ and that the equation of the tangent plane is $x + y + z = 3$.

Arfken 1.6.4 Find the total differential of a vector function $\vec{F}$.

One way to do this is to use index notation. Note that $\vec{F}$ is a vector and we represent it as F_i. Taking the total differential we get (and remembering that $\vec{F}$ depends on t as well as x, y, and z)

$$\begin{aligned} dF_i &= \partial_j F_i \cdot dx_j + \partial_t F_i \cdot dt \\ &= dx_j \cdot \partial_j F_i + \partial_t F_i \cdot dt \end{aligned}$$

Returning to vector notation, this is just

$$d\vec{F} = \left(d\vec{r} \cdot \vec{\nabla}\right) \vec{F} + \partial_t \vec{F} \cdot dt$$

Arfken 1.7.6 Find the divergence of the electric field of a point charge.

$$\begin{aligned} \nabla \cdot \vec{E} &= \frac{q}{4\pi\epsilon_0} \nabla \cdot \left(\frac{\vec{r}}{r^3}\right) \\ &= \frac{q}{4\pi\epsilon_0} \left[(\nabla \cdot \vec{r}) \frac{1}{r^3} + \vec{r} \cdot \nabla \left(\frac{1}{r^3}\right)\right] \\ &= \frac{q}{4\pi\epsilon_0} \left[3 \cdot \frac{1}{r^3} + \vec{r} \cdot (-3) \frac{\hat{\imath}x + \hat{\jmath}y + \hat{k}z}{r^5}\right] \\ &= \frac{3q}{4\pi\epsilon_0} \frac{1}{r^3} \left[1 - \frac{\vec{r} \cdot \vec{r}}{r^2}\right] \\ &= 0 \end{aligned}$$

where this argument is valid *only* for $r \neq 0$ for at $r = 0$, these expressions are undefined (and we have to use Dirac δ-functions).

Arfken 1.8.2 Show $\vec{u} \times \vec{v}$ is solenoidal if $\nabla \times \vec{u} = \nabla \times \vec{v} = 0$

One way to do this is to use index notation:

$$\begin{aligned} \partial_i \left(\epsilon_{ijk}\, u_j\, v_k\right) &= \epsilon_{ijk} \left(\partial_i u_j\, v_k + u_j\, \partial_i v_k\right) \\ &= v_k\, \epsilon_{kij} \partial_i u_j - u_j\, \epsilon_{jik} \partial_i v_k \end{aligned}$$

where, returning to vector notation, the last two terms include $\nabla \times \vec{u}$ and $\nabla \times \vec{v}$ which are, by assumption, zero. Therefore, the whole expression is zero as desired.

Arfken 1.9.1 Show $\vec{\nabla} \times (\vec{\nabla} \times \vec{V}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{V}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{V}$.

We will do this two ways. First, using components, let $\vec{A} = \nabla \times \vec{V}$ then we have

$$\vec{A} = \hat{\imath} \left(\partial_y V_z - \partial_z V_y\right) + \hat{\jmath} \left(\partial_z V_x - \partial_x V_z\right) + \hat{k} \left(\partial_x V_y - \partial_y V_x\right)$$

Now evaluate $\nabla \times (\nabla \times \vec{V})$:

$$\begin{aligned} \nabla \times (\nabla \times \vec{V}) &= \nabla \times \vec{A} \\ &= \hat{\imath} \left(\partial_y A_z - \partial_z A_y\right) + \hat{\jmath} \left(\partial_z A_x - \partial_x A_z\right) + \hat{k} \left(\partial_x A_y - \partial_y A_x\right) \\ &= \hat{\imath} \left[\partial_y \left(\partial_x V_y - \partial_y V_x\right) - \partial_z \left(\partial_z V_x - \partial_x V_z\right)\right] \\ &\quad \hat{\jmath} \left[\partial_z \left(\partial_y V_z - \partial_z V_y\right) - \partial_x \left(\partial_x V_y - \partial_y V_x\right)\right] \\ &\quad \hat{k} \left[\partial_x \left(\partial_z V_x - \partial_x V_z\right) - \partial_y \left(\partial_y V_z - \partial_z V_y\right)\right] \\ &= \hat{\imath} \left[\partial_x \partial_y V_y + \partial_x \partial_z V_z - \partial_y \partial_y V_x - \partial_z \partial_z V_x\right] \\ &\quad \hat{\jmath} \left[\partial_y \partial_x V_x + \partial_y \partial_z V_z - \partial_x \partial_x V_y - \partial_z \partial_z V_y\right] \\ &\quad \hat{k} \left[\partial_z \partial_x V_x + \partial_z \partial_y V_y - \partial_x \partial_x V_z - \partial_y \partial_y V_z\right] \\ &= \hat{\imath} \left[\partial_x \left(\nabla \cdot \vec{V}\right) - \nabla^2 V_x\right] + \hat{\jmath} \left[\partial_y \left(\nabla \cdot \vec{V}\right) - \nabla^2 V_y\right] + \hat{k} \left[\partial_z \left(\nabla \cdot \vec{V}\right) - \nabla^2 V_z\right] \\ &= \nabla \left(\nabla \cdot \vec{V}\right) - \nabla^2 (\vec{V}) \end{aligned}$$

Or, doing it in index notation, we get

$$\begin{aligned}\epsilon_{ijk}\,\partial_j\left(\epsilon_{klm}\,\partial_l V_m\right) &= \epsilon_{kij}\,\epsilon_{klm}\,\partial_j\left(\partial_l V_m\right)\\ &= \left(\delta_{il}\delta_{jm}-\delta_{im}\delta_{jl}\right)\partial_j\partial_l V_m\\ &= \partial_j\partial_i V_j - \partial_j\partial_j V_i\\ &= \partial_i\left(\partial_j V_j\right) - \left(\partial_j\partial_j\right)V_i\end{aligned}$$

where we have, in the last line, commuted partial derivatives. Converting to the usual vector notation, this is, of course, exactly the above expression and what we wanted to show.

Arfken 1.9.11 Show $(-i\vec{\nabla}-e\vec{A})\times(-i\vec{\nabla}-e\vec{A}_\psi)$ reduces to $ie\vec{B}\psi$ and other stuff.

Using index notation, we get

$$\begin{aligned}\epsilon_{ijk}\left(-i\partial_j - eA_j\right)\left(-i\partial_k\psi - eA_k\psi\right) &= \epsilon_{ijk}\left(-\partial_j\partial_k\psi + ieA_j\partial_k\psi + ie\partial_j\left(A_k\psi\right) + e^2A_jA_k\psi\right)\\ &= ie\,\epsilon_{ijk}\left(A_j\partial_k\psi + \partial_jA_k\cdot\psi + A_k\partial_j\psi\right)\\ &= ie\,\epsilon_{ijk}\,\partial_jA_k\psi\end{aligned}$$

where in going from the first to the second line two terms drop out because they are symmetric and are multiplying an antisymmetric object, ϵ_{ijk}. Another way to look at this is to think of the vector version as a vector crossed into itself. Going from the second to the third line we can cancel the two terms involving the derivative of ψ. Rewriting in the usual vector notation, the last term becomes just $ie\nabla\times\vec{A}\psi = ie\vec{B}\psi$.

The final part of the problem asks (poorly) for something about the g-factor. What this refers to is a standard problem in quantum mechanics. In particular, consider a charged particle in a (constant) magnetic field, $\vec{B}$. In this case, we must write the Hamiltonian using the above form for the generalized momentum:

$$H\psi = \left[\frac{1}{2m}\left(\vec{p}-e\vec{A}\right)\cdot\left(\vec{p}-e\vec{A}\right)+V\right]\psi$$

where we will ignore the V from now on. Our primary interest is in the non-kinetic energy terms that involve the magnetic vector potential, $\vec{A}$ and that are linear in the charge, e. So, we can focus on these terms:

$$\begin{aligned}-\frac{e}{2m}\left(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}\right)\psi &= \frac{ie}{2m}\left[\nabla\cdot(\vec{A}\psi)+\vec{A}\cdot\nabla\psi\right]\\ &= \frac{ie}{2m}\left[(\nabla\cdot\vec{A})\psi + 2\vec{A}\cdot\nabla\psi\right]\end{aligned}$$

For the form of $\vec{A}$ given in exercise 1.13.7, we have that $\vec{A}=\vec{B}\times\vec{r}/2$. We can show that for this choice of $\vec{A}$,

$$\begin{aligned}\nabla\cdot\vec{A} &= \frac{1}{2}\nabla\cdot\left(\vec{B}\times\vec{r}\right)\\ &= \frac{1}{2}\vec{r}\cdot\nabla\times\vec{B} - \frac{1}{2}\vec{B}\cdot\nabla\times\vec{r}\\ &= 0\end{aligned}$$

because $\vec{B}$ is constant and $\nabla\times\vec{r}=0$. What remains in our above expression, then, is (in index notation)

$$\begin{aligned}2A_i\partial_i\psi &= \epsilon_{ijk}B_jx_k\partial_i\psi\\ &= B_j\epsilon_{jki}x_k\partial_i\psi\end{aligned}$$

which is just $\vec{B}\cdot(\vec{r}\times\nabla\psi)$. Our final expression is then

$$-\frac{e}{2m}\left(\vec{p}\cdot\vec{A}+\vec{A}\cdot\vec{p}\right)\psi = -\frac{e}{2m}\,\vec{B}\cdot\vec{L}\psi$$

where we have used $\vec{L}=-i\vec{r}\times\nabla$. This last term can be interpreted as the interaction energy between a (charged, massive) particle with angular momentum and an external (constant) magnetic field. Because of the classical gyromagnetic ratio, we have the following relation for such a particle's magnetic dipole moment:

$$\vec{\mu} = g_L\,\frac{e}{2m}\,\vec{L}$$

where $g_L=1$ for classical electrodynamics. We can write our interaction energy then as $-g_L^{-1}\,\vec{B}\cdot\vec{\mu}\psi$. Note that in this problem, our units have been set to $c=\hbar=1$ so a Bohr magneton is $\mu_B=e/2m$ in these units.