Assignment 2

Arfken 1.10.2 Find the work done against the force $\vec{F} = (-\hat{x}y + \hat{y}x) / \left(x^2 + y^2\right)$ going around a unit circle halfway.

$$W = -\int \vec{F} \cdot d\vec{r}$$

Note that the unit vector here points in a purely angular direction: $d\vec{r} = \hat{\theta} d\theta$ and that the force can be written $\vec{F} = -\hat{x}\sin\theta + \hat{y}\cos\theta = \hat{\theta}$. So our integral becomes

$$\begin{aligned} W &= -\int_a^b \hat{\theta} \cdot \hat{\theta}\, d\theta \\ &= -\theta\Big|_a^b \\ &= -(b-a) \end{aligned}$$

where the limits for part (a) are from 0 to π and in (b) are from 0 to $-\pi$. Thus the work in the two cases is $-\pi$ and π respectively and the work is path dependent.

In section 1.13, there is a discussion on conservative forces and conditions necessary to have them. One is the vanishing of the line integral of the force around any closed path in our region (in this case, three dimensional Euclidean space). Said in an equivalent, but slightly different way, the line integral of the force between two points in the space must not depend on the particular path taken between those two points. Here it clearly does. So we say that the force is non-conservative. However, note that on taking the curl of the force field, we get $\nabla \times \vec{F} = 0$ and the force seems to be conservative after all. What's going on? The answer lies in a subtle assumption about differentiation (the curl) here. The force field is not differentiable (well behaved) on the axis, i.e. $x = 0, y = 0$. To handle this location correctly, we have to use Dirac delta functions and the curl turns out not to vanish along the axis. It does vanish everywhere else, just not on the axis. Said another way, any line integral along a closed path that encircles the axis will not vanish. Thus this force really is nonconservative.

Arfken 1.10.6 Find the curl from the appropriate integral representation:

$$\lim_{d\tau\to 0} \frac{\int_s d\vec{\sigma} \times \vec{V}}{\int d\tau} = \nabla \times \vec{V}$$

Consider the integrand evaluated on the sides of an infinitesimal box of volume $dxdydz$:

$$\begin{aligned} d\vec{\sigma} \times \vec{V} &= da\, \hat{n} \times \vec{V} \\ &= dydz\, \hat{\imath} \times \left(\vec{V}|_{x+dx} - \vec{V}|_x\right) + dzdx\, \hat{\jmath} \times \left(\vec{V}|_{y+dy} - \vec{V}|_y\right) + dxdy\, \hat{k} \times \left(\vec{V}|_{z+dz} - \vec{V}|_z\right) \\ &= dydz\, \hat{\imath} \times \left(dx\, \partial_x \vec{V} + \cdots\right) + dzdx\, \hat{\jmath} \times \left(dy\, \partial_y \vec{V} + \cdots\right) + dxdy\, \hat{k} \times \left(dz\, \partial_z \vec{V} + \cdots\right) \\ &= dxdydz\, \left(\hat{\imath} \times \partial_x \vec{V} + \hat{\jmath} \times \partial_y \vec{V} + \hat{k} \times \partial_z \vec{V}\right) \end{aligned}$$

where we have neglected higher order terms in the expectation that we will take the limit in which they go to zero. Doing that now, we get,

$$\begin{aligned} \lim_{d\tau\to 0} \frac{\int_s d\vec{\sigma} \times \vec{V}}{\int d\tau} &= \hat{\imath} \times \partial_x \vec{V} + \hat{\jmath} \times \partial_y \vec{V} + \hat{k} \times \partial_z \vec{V} \\ &= \left(\hat{k}\partial_x V_y - \hat{\jmath}\partial_x V_z\right) + \left(\hat{\imath}\partial_y V_z - \hat{k}\partial_y V_x\right) + \left(\hat{\jmath}\partial_z V_x - \hat{\imath}\partial_z V_y\right) \\ &= \hat{\imath}\left(\partial_y V_z - \partial_z V_y\right) + \hat{\jmath}\left(\partial_z V_x - \partial_x V_z\right) + \hat{k}\left(\partial_x V_y - \partial_y V_x\right) \\ &= \nabla \times \vec{V} \end{aligned}$$

Arfken 1.11.4 If ψ satisfies Laplace's equation, show the surface integral of its normal derivative vanishes.
Using the divergence theorem, we get for the vector $\nabla\psi$,

$$\begin{aligned}\oint_S \nabla\psi \cdot \hat{n}\, da &= \int_V \nabla \cdot (\nabla\psi)\; dv \\ &= \int_V \nabla^2 \psi\, dv \\ &= 0\end{aligned}$$

Arfken 1.11.9 Determine the work needed to assemble a local, steady state system of currents and fields.

Using the vector identitiy

$$\nabla \cdot \left(\vec{a} \times \vec{b}\right) = \vec{b} \cdot (\nabla \times \vec{a}) - \vec{a} \cdot \left(\nabla \times \vec{b}\right)$$

we can write the work as the following (with $d\tau$ the volume element)

$$\begin{aligned}W &= \frac{1}{2}\int \vec{H} \cdot \vec{B}\, d\tau \\ &= \frac{1}{2}\int \vec{H} \cdot \left(\nabla \times \vec{A}\right)\; d\tau \\ &= \frac{1}{2}\int \left[\nabla \cdot \left(\vec{A} \times \vec{H}\right) + \vec{A} \cdot \nabla \times \vec{H}\right]\; d\tau \\ &= \frac{1}{2}\oint \left(\vec{A} \times \vec{H}\right) \cdot \hat{n}\, da + \frac{1}{2}\int \vec{A} \cdot \left(\nabla \times \vec{H}\right)\; d\tau\end{aligned}$$

where we have used the divergence theorem for the first term in the last line. If we argue that our system of charges and current are localized and that the surface integral is done at very large radius, this surface integral can be set to zero. Further, using the Maxwell equations with the displacement current zero, $\nabla \times \vec{H} = \vec{J}$ and we have

$$W = \int \vec{A} \cdot \vec{J}\, d\tau$$

Arfken 1.11.10 Prove the generalization of Green's theorem.

$$\begin{aligned}\int_V (v\mathcal{L}u - u\mathcal{L}v)\; d\tau &= \int (v\nabla \cdot (p\nabla u) + vqu - u\nabla \cdot (p\nabla v) - uqv)\; d\tau \\ &= \int (\nabla \cdot [vp\nabla u] - \nabla v \cdot p\nabla u - \nabla [up\nabla v] + \nabla u \cdot p\nabla v)\; d\tau \\ &= \int \nabla \cdot [vp\nabla u - up\nabla v]\; d\tau \\ &= \oint_S p\,(v\nabla u - u\nabla v) \cdot \hat{n}\, da\end{aligned}$$

where in the second line we have canceled two terms and in the last line we have used the divergence theorem.

Arfken 1.12.5 Show the integral form of Ampere's law from its differential from.
Integrating one of Maxwell's equations, $\nabla \times \vec{H} = \vec{J}$, over a surface S, we get

$$\begin{aligned}\int_S \nabla \times \hat{H} \cdot d\vec{a} &= \int_S \vec{J} \cdot d\vec{a} \\ \oint_C \vec{H} \cdot d\vec{r} &= I\end{aligned}$$

where we have used Stokes' theorem and I is the enclosed current.

Arfken 1.13.3 Examine the force of gravity within a spherical, massive body.

Using the Poisson equation for a gravitational potential from section 1.14, $\nabla^2\phi = 4\pi G\rho$, and the fact that $\vec{F} = -\vec{\nabla}\phi$, we can write down a gravitational analog of Gauss' law:

$$\int \vec{F}\cdot\hat{n}da = \int -4\pi G\rho\, dv$$

Evaluating this on a spherical gaussian surface *within* a massive, spherical body with constant density, ρ_0, and radius a, we get

$$\begin{aligned} -|\vec{F}|\, 4\pi r^2 &= -4\pi G\rho_0\, \frac{4}{3}\pi r^3 \\ \vec{F} &= -\frac{4}{3}\pi G\rho_0\, r\hat{r} \end{aligned}$$

where we have assumed that the direction of $\vec{F}$ was opposite that of the normal, $\hat{n}$, to the gaussian surface of radius $r < a$. Finally, we note that this is the force on a unit mass, m_0.

The gravitational potential which corresponds to this force is

$$\begin{aligned} \phi &= -\int_0^r \vec{F}\cdot d\vec{r} \\ &= -\int_0^r -\frac{4}{3}\pi G\rho_0 r'\hat{r}'\cdot\hat{r}'\, dr' \\ &= \frac{4}{3}\pi G\rho_0\frac{r^2}{2} \end{aligned}$$

Finally, the equation of motion for a particle traversing the inside of the earth assuming it has constant density as above is

$$\vec{F} = \ddot{\vec{r}} = -\frac{4\pi G\rho_0}{3}\vec{r}$$

where, again, we assume that our particle has unit mass, $m_0 = 1$. This, of course, is the equation for simple harmonic motion with frequency $\sqrt{\frac{4\pi G\rho_0}{3}}$. Putting appropriate numerical values in here, we get 2π over this, the period, to be about 5070 seconds, not quite an hour and a half.

Arfken 1.13.4 Find the potential from the given tidal force:

$$\vec{F} = -GMm\frac{1}{R^3}(x, y, -2z)$$

The potential is, assuming that R is essentially a constant,

$$\begin{aligned} \phi &= -\int \vec{F}\cdot d\vec{r} \\ &= \frac{GMm}{R^3}\int xdx + ydy - 2zdz \\ &= \frac{GMm}{R^3}\frac{1}{2}(x^2 + y^2 - 2z^2) \end{aligned}$$

Arfken 1.13.5 Find a vector potential, $\vec{A}$, for a magnetic induction B given by $\frac{\mu_0 I}{2\pi}(-y, z, 0)/(x^2+y^2)$

The equations are for $\vec{B} = \nabla\times\vec{A}$:

$$\begin{aligned} B_x &= \partial_y A_z - \partial_z A_y = \frac{\mu_0 I}{2\pi}\frac{-y}{x^2+y^2} \\ B_y &= \partial_z A_x - \partial_x A_z = \frac{\mu_0 I}{2\pi}\frac{x}{x^2+y^2} \\ B_z &= \partial_x A_y - \partial_y A_x = 0 \end{aligned}$$

As in the text and as we did in class, we will choose A_x to be zero. This seems odd, but we can do it because of the arbitrariness of the gauge potential $\vec{A}$. The B_z equation then implies that A_y is a function that depends only on y and z. At the same time we can integrate the B_y equation with respect to x:

$$\begin{aligned}\partial_x A_z &= -\frac{\mu_0 I}{2\pi}\frac{x}{x^2+y^2} \\ A_z &= -\frac{\mu_0 I}{2\pi}\int \frac{x}{x^2+y^2}\,dx \\ &= -\frac{\mu_0 I}{2\pi}\frac{1}{2}\ln|x^2+y^2| + f(y,z)\end{aligned}$$

where f is an arbitrary function of y and z. Substituting this expression in the B_x equation, we get

$$\begin{aligned}\partial_y A_z - \partial_z A_y &= -\frac{\mu_0 I}{2\pi}\frac{y}{x^2+y^2} \\ -\frac{\mu_0 I}{2\pi}\frac{1}{2}\frac{2y}{x^2+y^2} + \partial_y f(y,z) - \partial_z A_y &= -\frac{\mu_0 I}{2\pi}\frac{y}{x^2+y^2}\end{aligned}$$

leaving us with

$$\partial_z A_y - \partial_y f(y,z) = 0$$

which can be integrated to give

$$A_y = \int \partial_y f(y,z)dz + g(y)$$

where both f and g are arbitrary functions of their arguments. We can witout loss of generality choose $f = g = 0$ and our final solution is then $A_x = A_y = 0$ and

$$A_z = -\frac{\mu_0 I}{4\pi}\ln|x^2+y^2|$$

Arfken 1.14.1 Find a two-dimensional Gauss' law.

Recall that we originally developed the divergence theorem in class in its two-dimensional form and showed that it extended to three and other dimensions. We will use that here as well as our derivation of the usual Gauss' law from class and the text.

Imagine a two-dimensional surface, S, bounded by a contour, C. For any vector field, $\vec{V}$, on that surface, the divergence theorem in two dimensions is

$$\int_S \nabla\cdot\vec{V}da = \oint_C \vec{V}\cdot\hat{n}\,dl$$

where $\hat{n}$ is the outward normal to the contour and dl is the integration variable along the contour, C. Now let $\vec{V} = \vec{E} = \frac{q}{2\pi\epsilon_0}\frac{\hat{\rho}}{\rho}$, the electric field of a point charge, q, placed at the origin in this two-dimensional space. If our contour, C, does *not* enclose the origin, the integrals above are zero since we can show that

$$\nabla\cdot\vec{E} \propto \nabla\cdot\left(\frac{\hat{\rho}}{\rho}\right) = 0$$

However, if the contour, C, encloses the origin, we must make an argument similar to the one we made in class. Namely, let C be composed of two contours, C_1 and C_2, with C_1 being the "outer" contour and C_2 a small circular contour surrounding the origin and connected to C_1 by a thin "passageway" whose thickness will be considered negligible (and whose contributions to the integral will ultimately cancel). Since the contour C does not actually include the origin the integral of $\vec{E}$ around it will be zero by our earlier argument. Thus we have

$$\begin{aligned} 0 &= \frac{q}{2\pi\epsilon_0} \oint_{C_1+C_2} \frac{\hat{\rho}}{\rho} \cdot \hat{n}\, dl \\ &= \frac{q}{2\pi\epsilon_0} \oint_{C_1} \frac{\hat{\rho}}{\rho} \cdot \hat{n}\, dl + \frac{q}{2\pi\epsilon_0} \int_0^{2\pi} \frac{\hat{\rho}}{\delta} \cdot (-\hat{\rho})\ \delta d\theta \\ &= \frac{q}{2\pi\epsilon_0} \oint_{C_1} \frac{\hat{\rho}}{\rho} \cdot \hat{n}\, dl - \frac{q}{\epsilon_0} \end{aligned}$$

where we have assumed that the contour C_2 around the origin has a radius of δ with the normal, $\hat{n}$, pointed inward towards the origin (*out* of the contour of interest).

Putting all of this together, we see that

$$\oint_C \vec{E} \cdot \hat{n}\, dl = \frac{q}{\epsilon_0}$$

if C encloses charge and zero otherwise.

Arfken 1.15.9 Evaluate the derivative of the delta function.

$$\begin{aligned} \int_{-\infty}^{\infty} \delta'(x) f(x)\, dx &= \delta(x) f(x) \Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x) f'(x)\, dx \\ &= -f'(0) \end{aligned}$$

where we have used the assumed properties of $f(x)$, namely that it is smooth and drops to zero at $\pm\infty$ to eliminate the first term. Similarly, we can argue that except at $x = 0$, $\delta(x) = 0$.