Assignment 3

Arfken 2.1.6 The metric or spacetime interval in Minkowski space is

$$ds^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2$$

We can read off the coefficients g_{ij} as

$$g_{00} = 1 \qquad g_{11} = g_{22} = g_{33} = -1$$

with all the rest (off-diagonal terms) being zero. Putting this into a matrix form, we get

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Arfken 2.2.2 Find the divergence and curl of the unit vector $\hat{e}_1$ in an arbitrary orthogonal coordinate system.

Using equation 2.21 with $\vec{V} = (1, 0, 0)$ we get for the divergence

$$\vec{\nabla} \cdot \vec{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1} (h_2 h_3) \right]$$

with the other terms zero.

Using the determinant form for the curl in equation 2.27, we get

$$\vec{\nabla} \times \vec{V}(q_1, q_2, q_3) = \frac{1}{h_1 h_2 h_3} \left[\hat{e}_2 \, h_2 \frac{\partial h_1}{\partial q_3} - \hat{e}_3 \, h_3 \frac{\partial h_1}{\partial q_2} \right]$$

Arfken 2.4.11 Show that $(\vec{\nabla} \cdot \vec{\nabla})(\vec{\nabla} \times \vec{v}) = 0$ with $\vec{v} = \hat{z}\, v(\rho)$ leads to a third order differential equation satisfied by $v = v_0 + a_2 \rho^2$.

First take the curl of our $\vec{v}$:

$$\vec{\nabla} \times \vec{v} = -\hat{\varphi} \frac{\partial v}{\partial \rho}$$

Now we must take the *vector* Laplacian of this vector, call it $\vec{V} = \vec{\nabla} \times \vec{v}$. Note that $V_\varphi = -\partial v / \partial \rho$ is the only nonzero component. Using Eq. 2.37 we get

$$\left((\vec{\nabla} \cdot \vec{\nabla}) \, \vec{V} \right) \Big|_\varphi = \nabla^2 V_\varphi - \frac{1}{\rho^2} V_\varphi$$

where we are taking the φ component of the vector Laplacian equation. Now we take the *scalar* Laplacian of the φ component of $\vec{V}$:

$$\begin{aligned} \left((\vec{\nabla} \cdot \vec{\nabla}) \, \vec{V} \right) \Big|_\varphi &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial V_\varphi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 V_\varphi}{\partial \varphi^2} + \frac{\partial^2 V_\varphi}{\partial z^2} - \frac{1}{\rho^2} V_\varphi \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(-\rho \frac{\partial^2 v}{\partial \rho^2} \right) - \frac{1}{\rho^2} \left(-\frac{\partial v}{\partial \rho} \right) \end{aligned}$$

Since this is zero by assumption, it is exactly the differential equation we want (up to a multiplication by -1). It is then straightforward to show that $v = v_0 + a_2 \rho^2$ satisfies it.

Arfken 2.4.14 Consider TEM waves in a coaxial wave guide.

The electric field $\vec{E} = \vec{E}(\rho, \varphi)e^{i(kz-\omega t)}$ and magnetic induction $\vec{B} = \vec{B}(\rho, \varphi)e^{i(kz-\omega t)}$ are such that $\vec{E}(\rho, \varphi)$ and $\vec{B}(\rho, \varphi)$ both satisfy the vector Laplacian equation. We want to show that $\vec{E}(\rho, \varphi) = \hat{\rho}E_0 a/\rho$ and $\vec{B}(\rho, \varphi) = \hat{\varphi}B_0 a/\rho$ satisfy their respective equations.

For $\vec{E}(\rho, \varphi)$, the relevant equation is just the ρ component of the vector Laplacian given in equation 2.35 since the other components vanish:

$$\begin{aligned}\left(\vec{\nabla} \cdot \vec{\nabla}\right) \vec{E} &= \nabla^2 E_\rho - \frac{1}{\rho^2} E_\rho \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial E_\rho}{\partial \rho} \right) - \frac{1}{\rho^2} E_\rho \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(-\frac{1}{\rho} E_0 a \right) - \frac{1}{\rho^2} \frac{E_0 a}{\rho} \\ &= 0\end{aligned}$$

For $\vec{B}(\rho, \varphi)$, the relevant equation is now the φ component of the vector Laplacian:

$$\left(\vec{\nabla} \cdot \vec{\nabla}\right) \vec{B} = \nabla^2 B_\varphi - \frac{1}{\rho^2} B_\varphi$$

and in an otherwise indentical calculation, this yields 0 for the above form for $B_\varphi(\rho, \varphi)$.

Now we must verify that the general solutions satisfy Maxwell's equations (for example look at the equations in the introduction of the text). The two divergence equations are

$$\begin{aligned}0 &= \vec{\nabla} \cdot \vec{E} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho E_\rho) + \frac{1}{\rho} \frac{\partial E_\varphi}{\partial \varphi} + \frac{\partial E_z}{\partial z} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (E_0 a) \\ &= 0 \\ 0 &= \vec{\nabla} \cdot \vec{B} \\ &= \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho B_\rho) + \frac{1}{\rho} \frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z} \\ &= 0\end{aligned}$$

The curl equations are

$$\begin{aligned}0 &= \nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \\ &= \frac{1}{\rho} \left(\rho \hat{\varphi}\, \partial_z E_\rho - \hat{z}\, \partial_\varphi E_\rho \right) + \frac{\partial \vec{B}}{\partial t} \\ &= \frac{1}{\rho} \left(\rho \hat{\varphi}\, ikE_0 \frac{a}{\rho} e^{i(kz-\omega t)} - 0 \right) + \hat{\varphi}(-i\omega) B_0 \frac{a}{\rho} e^{i(kz-\omega t)} \\ &= \hat{\varphi} \frac{i}{\rho} a e^{i(kz-\omega t)} (kE_0 - \omega B_0) \\ 0 &= \nabla \times \vec{B} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ &= \frac{1}{\rho} \left(\hat{z}\, \partial_\rho (\rho B_\varphi) - \hat{\rho}\, \partial_z (\rho B_\varphi) \right) - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \\ &= \frac{1}{\rho} \left(0 - \hat{\rho}\, ik\rho B_0 \frac{a}{\rho} e^{i(kz-\omega t)} \right) - \mu_0 \epsilon_0 (-i\omega) \hat{\rho} E_0 \frac{a}{\rho} e^{i(kz-\omega t)} \\ &= \hat{\rho}\, \frac{i}{\rho} a e^{i(kz-\omega t)} (-kB_0 + \mu_0 \epsilon_0 \omega E_0)\end{aligned}$$

This will be consistent provided $B_0/E_0 = k/\omega = \mu_0\epsilon_0\,\omega/k$ as demanded.

Arfken 2.4.15 For $\vec{B} = \hat{\varphi}B_\varphi(\rho)$, find $(\vec{B}\cdot\vec{\nabla})\,\vec{B}$

$$
\begin{aligned}
(\vec{B}\cdot\vec{\nabla})\,\vec{B} &= (B_\rho\partial_\rho + B_\varphi\frac{1}{\rho}\partial_\varphi + B_z\partial_z)\,\vec{B} \\
&= B_\varphi(\rho)\frac{1}{\rho}\,\partial_\varphi\;[\hat{\varphi}\,B_\varphi(\rho)] \\
&= \frac{{B_\varphi}^2}{\rho}\partial_\varphi\,(\hat{\varphi}) \\
&= -\hat{\rho}\frac{{B_\varphi}^2}{\rho}
\end{aligned}
$$

where in the first line $B_\rho = B_z = 0$ and we have used the result (*e.g.* from Exercise 2.4.3) $\partial_{\hat{\varphi}}\,(\varphi) = -\hat{\rho}$ in the last line.

Arfken 2.5.2 Find the partial derivatives of the unit vectors in spherical polar coordinates and use these to derive the Laplacian in these coordinates.

Using the expressions for $\hat{r}$, $\hat{\theta}$ and $\hat{\varphi}$ from exercise 2.5.1:

$$
\begin{aligned}
\hat{r} &= \quad \hat{x}\sin\theta\cos\varphi + \hat{y}\sin\theta\sin\varphi + \hat{z}\cos\theta \\
\hat{\theta} &= \quad \hat{x}\cos\theta\cos\varphi + \hat{y}\cos\theta\sin\varphi - \hat{z}\sin\theta \\
\hat{\varphi} &= -\hat{x}\sin\varphi + \hat{y}\cos\varphi
\end{aligned}
$$

From these, we see the following

$$\hat{r}_{,r} = \hat{\theta}_{,r} = \hat{\varphi}_{,r} = 0$$

The θ derivatives are

$$\hat{r}_{,\theta} = \hat{\theta} \qquad \hat{\theta}_{,\theta} = -\hat{r} \qquad \hat{\varphi}_{,\theta} = 0$$

and the φ derivatives are

$$\hat{r}_{,\varphi} = \sin\theta\,\hat{\varphi} \qquad \hat{\theta}_{,\varphi} = \cos\theta\,\hat{\varphi} \qquad \hat{\varphi}_{,\varphi} = -(\sin\theta\,\hat{r} + \cos\theta\,\hat{\theta})$$

We can now construct the scalar Laplacian $\vec{\nabla}\cdot\vec{\nabla}\psi$ in the following way

$$\vec{\nabla}\cdot\vec{\nabla}\psi = \left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial}{\partial\theta} + \hat{\varphi}\,\frac{1}{r\sin\theta}\frac{\partial}{\partial\varphi}\right)\cdot\left(\hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right)$$

such that when we apply the second $\vec{\nabla}$, we use the differential operators before finding the scalar product between the unit vectors. We get

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{\nabla}\psi = \quad & \hat{r}\cdot\left\{\hat{r}_{,r}\,\psi_{,r} + \hat{r}\psi_{,rr} + \hat{\theta}_{,r}\,\frac{1}{r}\psi_{,\theta} + \hat{\theta}\left(\frac{1}{r}\psi_{,\theta}\right)_{,r} + \hat{\varphi}_{,r}\,\frac{1}{r\sin\theta}\,\psi_{,\varphi} + \hat{\varphi}\left(\frac{1}{r\sin\theta}\,\psi_{,\varphi}\right)_{,r}\right\} \\
& + \hat{\theta}\cdot\,\frac{1}{r}\left\{\hat{r}_{,\theta}\,\psi_{,r} + \hat{r}\psi_{,\theta r} + \hat{\theta}_{,\theta}\,\frac{1}{r}\psi_{,\theta} + \hat{\theta}\frac{1}{r}\psi_{,\theta\theta} + \hat{\varphi}_{,\theta}\,\frac{1}{r\sin\theta}\,\psi_{,\varphi} + \hat{\varphi}\left(\frac{1}{r\sin\theta}\,\psi_{,\varphi}\right)_{,\theta}\right\} \\
& + \hat{\varphi}\cdot\,\frac{1}{r\sin\theta}\left\{\hat{r}_{,\varphi}\,\psi_{,r} + \hat{r}\psi_{,\varphi r} + \hat{\theta}_{,\varphi}\,\frac{1}{r}\psi_{,\theta} + \hat{\theta}\frac{1}{r}\psi_{,\varphi\theta} + \hat{\varphi}_{,\varphi}\,\frac{1}{r\sin\theta}\,\psi_{,\varphi} + \hat{\varphi}\left(\frac{1}{r\sin\theta}\,\psi_{,\varphi}\right)_{,\varphi}\right\} \\
& = \psi_{,rr} + \frac{1}{r}\,\psi_{,r} + \frac{1}{r}\,\psi_{,r} + \frac{\cot\theta}{r}\,\psi_{,\theta} + \frac{1}{r^2}\,\psi_{,\theta\theta} + \frac{1}{r^2\sin^2\theta}\,\psi_{,\varphi\varphi}
\end{aligned}
$$

which is indeed the Laplacian in spherical coordinates and where we have used the comma notation to denote partial differentiation.

Arfken 2.5.10 Find the spherical coordinate components of a particle moving through space with distance vector $\vec{r}(t) = \hat{r}(t)r(t)$

To do this, we need the time derivatives of the unit vectors:

$$\begin{aligned}
\dot{\hat{r}} &= \hat{x}\left(\cos\theta\cos\varphi\,\dot{\theta} - \sin\theta\sin\varphi\,\dot{\varphi}\right) \\
&\quad + \hat{y}\left(\cos\theta\sin\varphi\,\dot{\theta} + \sin\theta\cos\varphi\,\dot{\varphi}\right) \\
&\quad - \hat{z}\,\sin\theta\,\dot{\theta} \\
&= \hat{\theta}\,\dot{\theta} + \hat{\varphi}\,\dot{\varphi}\sin\theta \\
\dot{\hat{\theta}} &= \hat{x}\left(-\sin\theta\cos\varphi\,\dot{\theta} - \cos\theta\sin\varphi\,\dot{\varphi}\right) \\
&\quad + \hat{y}\left(-\sin\theta\sin\varphi\,\dot{\theta} + \cos\theta\cos\varphi\dot{\varphi}\right) \\
&\quad - \hat{z}\,\cos\theta\dot{\theta} \\
&= -\hat{r}\,\dot{\theta} + \hat{\varphi}\,\cos\theta\dot{\varphi} \\
\dot{\hat{\varphi}} &= -\hat{x}\cos\varphi\,\dot{\varphi} - \hat{y}\sin\varphi\,\dot{\varphi} \\
&= -\hat{r}\,\sin\theta\dot{\varphi} - \hat{\theta}\,\cos\theta\dot{\varphi}
\end{aligned}$$

The velocity vector is $\vec{v} = \dot{\vec{r}}(t)$:

$$\begin{aligned}
\dot{\vec{r}} &= \hat{r}(t)\,\dot{r}(t) + \dot{\hat{r}}\,r(t) \\
&= \dot{r}\,\hat{r} + r\left(\dot{\theta}\,\hat{\theta} + \sin\theta\dot{\varphi}\,\hat{\varphi}\right)
\end{aligned}$$

The acceleration vector is $\vec{a} = \ddot{\vec{r}}(t)$:

$$\begin{aligned}
\ddot{\vec{r}} &= \ddot{r}\hat{r} + \dot{r}\dot{\hat{r}} + \dot{r}\left(\dot{\theta}\,\hat{\theta} + \dot{\varphi}\sin\theta\,\hat{\varphi}\right) + r\left(\ddot{\theta}\,\hat{\theta} + \dot{\theta}\dot{\hat{\theta}} + \ddot{\varphi}\sin\theta\hat{\varphi} + \dot{\varphi}\cos\theta\dot{\theta}\,\hat{\varphi} + \dot{\varphi}\sin\theta\,\dot{\hat{\varphi}}\right) \\
&= \ddot{r}\,\hat{r} + \dot{r}\left[\dot{\theta}\hat{\theta} + \hat{\varphi}\dot{\varphi}\sin\theta\right] + \dot{r}\left(\dot{\theta}\hat{\theta} + \dot{\varphi}\sin\theta\,\hat{\varphi}\right) \\
&\quad + r\left[\ddot{\theta}\,\hat{\theta} + \dot{\theta}\left(-\hat{r}\dot{\theta} + \cos\theta\dot{\varphi}\hat{\varphi}\right) + \ddot{\varphi}\sin\theta\hat{\varphi} + \dot{\varphi}\cos\theta\dot{\theta}\hat{\varphi} - \dot{\varphi}^2\sin\theta\left(\hat{r}\sin\theta + \hat{\theta}\cos\theta\right)\right] \\
&= \hat{r}\left[\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\varphi}^2\right] + \hat{\theta}\left[2\dot{r}\dot{\theta} + r\ddot{\theta} - \dot{\varphi}^2\sin\theta\cos\theta\right] + \hat{\varphi}\left[2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta + r\ddot{\varphi}\sin\theta\right]
\end{aligned}$$

and for both, one can read off the appropriate components.

Arfken 2.5.11 From Newton's second law, $m\,\ddot{\vec{r}} = \hat{r}\,f(\vec{r})$, show $\vec{r}\times\dot{\vec{r}} = \vec{c}$

Cross $\vec{r}$ into the second law to get

$$m\,\vec{r}\times\ddot{\vec{r}} = \vec{r}\times\hat{r}\,f(\vec{r}) = 0$$

and notice that

$$\begin{aligned}
\frac{d}{dt}\left(\vec{r}\times\dot{\vec{r}}\right) &= \dot{\vec{r}}\times\dot{\vec{r}} + \vec{r}\times\ddot{\vec{r}} \\
&= \vec{r}\times\ddot{\vec{r}} \\
&= 0
\end{aligned}$$

Integrating this then leads to the desired result $\vec{r}\times\dot{\vec{r}} = \vec{c}$ where $\vec{c}$ is a constant vector. This is nothing more than a statement that angular momentum is conserved (*i.e.* a constant: $m\vec{c}$). Geometrically, one can interpret this as the area of the parallelogram formed by $\vec{r}$ and $\dot{\vec{r}}$, *i.e.* the rate at which the radius vector sweeps out area is a constant.

Arfken 2.5.17 Verify some operator identities with $\vec{L} = -i\vec{r} \times \vec{\nabla}$.

$$\begin{aligned}
-i\vec{r} \times \vec{L}\psi &= -\vec{r} \times (\vec{r} \times \nabla\psi) \\
&= -\vec{r} \times \left(\hat{\varphi}\,\frac{\partial\psi}{\partial\theta} - \hat{\theta}\,\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\varphi}\right) \\
&= \hat{\theta}\,r\frac{\partial\psi}{\partial\theta} + \hat{\varphi}\,\frac{r}{\sin\theta}\frac{\partial\psi}{\partial\varphi} \\
&= r^2\left[\hat{r}\frac{\partial\psi}{\partial r} + \hat{\theta}\,\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \hat{\varphi}\,\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\varphi}\right] - \hat{r}\,r^2\frac{\partial\psi}{\partial r} \\
&= r^2\nabla\psi - \hat{r}\,r^2\frac{\partial\psi}{\partial r}
\end{aligned}$$

Rearranging this and peeling off the ψ gives us

$$\nabla = \hat{r}\,\frac{\partial}{\partial r} - i\frac{\vec{r} \times \vec{L}}{r^2}$$

Now consider

$$\begin{aligned}
i\,\nabla \times \vec{L}\psi &= \nabla \times (\vec{r} \times \nabla\psi) \\
&= \nabla \times \left(\hat{\varphi}\,\psi_{,\theta} - \hat{\theta}\,\frac{1}{\sin\theta}\psi_{,\varphi}\right) \\
&= \frac{1}{r^2\sin\theta}\left\{\hat{r}\left(r\sin\theta\psi_{,\theta}\right)_{,\theta} + \hat{r}\left(\frac{r}{\sin\theta}\psi_{,\varphi}\right)_{,\varphi} - r\hat{\theta}\,\left(r\sin\theta\psi_{,\theta}\right)_{,r} + r\sin\theta\hat{\varphi}\left(-\frac{r}{\sin\theta}\psi_{,\varphi}\right)_{,r}\right\} \\
&= \vec{r}\left[\frac{1}{r^2}\left(r^2\psi_{,r}\right)_{,r} + \frac{1}{r^2\sin\theta}\left(\sin\theta\psi_{,\theta}\right)_{,\theta} + \frac{1}{r^2\sin^2\theta}\psi_{,\varphi\varphi}\right] \\
&\qquad - \frac{\hat{r}}{r}\left(r^2\psi_{,r}\right)_{,r} - \frac{1}{r}\hat{\theta}\left(r\psi_{,\theta}\right)_{,r} - \hat{\varphi}\frac{1}{r\sin\theta}\left(r\psi_{,\varphi}\right)_{,r} \\
&= \vec{r}\,\nabla^2\psi - [\hat{r}\left(\psi_{,r} + \psi_{,r} + r\psi_{,rr}\right) \\
&\qquad\quad + \hat{\theta}\frac{1}{r}\,\left(\psi_{,\theta} + r\psi_{,r\theta}\right) \\
&\qquad\quad + \hat{\varphi}\frac{1}{r\sin\theta}\,\left(\psi_{,\varphi} + r\psi_{,r\varphi}\right)\,] \\
&= \vec{r}\,\nabla^2\psi - \left[\nabla\psi + \vec{\nabla}\left(r\psi_{,r}\right)\right] \\
&= \vec{r}\,\nabla^2\psi - \vec{\nabla}\left(1 + r\frac{\partial}{\partial r}\right)\psi
\end{aligned}$$